

# SYMPLECTIC TORIC STRATIFIED SPACES WITH ISOLATED SINGULARITIES

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**ABSTRACT.** The goal of this paper is to classify symplectic toric stratified spaces with isolated singularities. This extends a result of Burns, Guillemin, and Lerman which carries out this classification in the compact connected case. On the way to this classification, we also classify symplectic toric cones. Via a well-known equivalence between symplectic toric cones and contact toric manifolds, this allows for the classification of contact toric manifolds as well, extending Lerman's classification of compact connected contact toric manifolds.

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## 1. INTRODUCTION

Symplectic geometry has a recent history of classification results associated to symplectic toric objects. In 1988, Delzant classified compact connected symplectic toric manifolds by the images of their moment maps [7]. Recently, this was extended by Karshon and Lerman to non-compact symplectic toric manifolds [13]. Research in this area has been dominated by two separate pursuits: examining what happens when the symplectic structure is weakened (see origami/folded symplectic manifolds [6], and  $b$ -symplectic/log symplectic manifolds [9]/[8]) and considering weakened versions of manifolds (see symplectic toric orbifolds [17], for instance). The goal of this paper, the classification of toric symplectic stratified spaces with isolated singularities, follows the latter trend.

The importance of stratified spaces in symplectic geometry arises from the symplectic reduction of Marsden-Weinstein [18] and Meyer [19]. In 1991, Sjamaar and Lerman [23] showed that, in general, symplectic reduction results in a stratified space and, furthermore, that each strata inherits a symplectic form from the original manifold. In 2005, Burns, Guillemin, and Lerman [5] defined symplectic toric stratified spaces with isolated singularities and classified these in the compact connected case using the images of their moment maps.

The foundation for Delzant's classification are the convexity and connectedness theorems of Atiyah [1] and Guillemin-Sternberg [10]. This is emulated by Burns, Guillemin, and Lerman who use a similar convexity and connectedness theorem for compact stratified spaces. The issue with the non-compact version of either case is that the image of the moment map no longer needs to be convex and its fibers need not be connected.

Karshon and Lerman's solution to this problem in the case of a symplectic toric manifold  $(M, \omega, \mu)$  is to substitute for the moment map image the orbital moment map: the unique map  $\bar{\mu}$  from the quotient of  $M$  to the Lie algebra dual through which  $\mu$  factors. This extra information supplements the loss of connected fibers. As the quotient of  $M$  by the torus action needn't be contractible, multiple isomorphism classes may be associated to each orbital moment map and these classes are quantified by cohomology classes of the quotient of  $M$ . Our classification will follow this approach.

Fix  $G$  a torus and let  $\mathfrak{g}$  denote its Lie algebra. A *symplectic toric stratified space with isolated singularities*  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  is (roughly) defined as a symplectic toric manifold with isolated singularities whose deleted neighborhoods are modeled on symplectic toric cones. Here,  $X$  is the full space,  $\omega$  is a symplectic form on  $X_{\text{reg}}$ , the open, dense manifold in  $X$ , and  $\mu$  is a continuous function such that  $\mu|_{X_{\text{reg}}}$  is a moment map for the action of  $G$  on  $(X_{\text{reg}}, \omega)$ .

By identifying the orbital moment maps of symplectic toric stratified spaces with isolated singularities as a type of map we call *stratified unimodular local embeddings*, we show that, by grouping together symplectic toric stratified spaces by these orbital moment map types, we can make the following classification:

**Theorem 1.1.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then there is a subspace  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$  so that the isomorphism classes of symplectic toric stratified spaces with isolated singularities  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  with  $G$ -quotient map  $\pi : X \rightarrow W$  and orbital moment map  $\psi$  are in natural bijective correspondence with the cohomology classes

$$H^2(W_{\text{reg}}, \mathbb{Z}_G) \times \mathcal{C}$$

where  $\mathbb{Z}_G$  is the integral lattice of  $G$ , the kernel of the map  $\exp : \mathfrak{g} \rightarrow G$ .

Once the relevant language has been established, the subspace  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$  is easily described and may be calculated through the use of relative de Rham cohomology.

On the way to making this classification, we will find it necessary to completely understand symplectic toric cones. Recall that a symplectic toric manifold  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  is a *symplectic toric cone* if  $M$  has a free and proper action of  $\mathbb{R}$  commuting with the action of  $G$  and, with respect to any action diffeomorphism  $\rho_\lambda : M \rightarrow M$  for this  $\mathbb{R}$  action (for  $\lambda \in \mathbb{R}$ ), we have  $\rho_\lambda^* \omega = e^\lambda \omega$ . Additionally, we impose that the moment map  $\mu$  for  $M$  is the *homogeneous moment map* for  $(M, \omega)$ , satisfying  $\mu(t \cdot p) = e^t \mu(p)$  for every  $t \in \mathbb{R}$  and  $p \in M$  (such a moment map for  $(M, \omega)$  always exists).

As in the case of symplectic toric stratified spaces, the orbital moment maps of symplectic toric cones must take a certain form: that of a **homogeneous unimodular local embedding**. We may group our symplectic toric cones by orbital moment map type to make the classification:

**Theorem 1.2.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then the set of isomorphism classes of symplectic toric cones  $(M, \omega, \mu)$  with  $G$ -quotient  $\pi : M \rightarrow W$  and orbital moment map  $\psi$  is in natural bijective correspondence with the cohomology classes  $H^2(W, \mathbb{Z}_G)$ , where  $\mathbb{Z}_G$  is the integral lattice of  $G$ , the kernel of the map  $\exp : \mathfrak{g} \rightarrow G$ .

As symplectic toric cones and contact toric manifolds are intimately related (indeed, they form equivalent categories), the classification of Theorem 1.2 allows us to classify contact toric manifolds:

**Theorem 1.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then there is a natural bijective correspondence between the set of isomorphism classes of contact toric manifolds  $(B, \xi)$  with symplectizations  $\xi_+^o$  admitting a  $G$ -quotient map  $\pi : \xi_+^o \rightarrow W$  with orbital moment map  $\psi$  and the cohomology classes  $H^2(W, \mathbb{Z}_G)$ .

This classification extends a classification of Lerman [15] in the case of compact connected contact toric manifolds.

The paper is organized as follows. In Section 2, we give a brief review of the classification result of Karshon and Lerman [13]. This will serve as a model of the techniques the rest of this paper will use as well as a repository for the results from this classification we will be adapting. The remainder of the paper is split into two main parts: Part I, which deals with the classification of symplectic toric cones and Part II, which deals with the classification of symplectic toric stratified spaces with isolated singularities. Each part begins with its own introduction and organizational description. Finally, there is a three part appendix, dealing with the basics of symplectic cones and contact manifolds, stacks, and relative de Rham cohomology.

**Notation and Conventions:** Manifolds are assumed to be finite dimensional, paracompact, and Hausdorff. For any action of a group  $K$  on a manifold  $M$  and for any point  $p$  of  $M$ ,  $K_p$  will denote the stabilizer of  $p$  in  $K$ .  $G$  will always denote a torus (a compact connected commutative finite dimensional Lie group) and  $\mathfrak{g}$  will always denote its Lie algebra.  $\mathbb{Z}_G$  will always be used to denote the integral lattice of  $\mathfrak{g}$ ; that is, the lattice  $\ker(\exp : \mathfrak{g} \rightarrow G)$ . The notation  $\langle \cdot, \cdot \rangle$  will denote the canonical pairing  $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ .

In using manifolds with corners, we will follow Karshon and Lerman and use the convention of Joyce (see [12]). An  $n$ -dimensional manifold with corners is a paracompact Hausdorff topological spaces admitting an atlas of charts to sectors of  $\mathbb{R}^n$  (open subsets of subsets of

$\mathbb{R}^n$  of the form  $[0, \infty)^k \times \mathbb{R}^{n-k}$ ). Smooth maps of manifolds with corners are then defined in the same way smooth maps of manifolds are defined: continuous maps that locally factor through smooth maps on charts. In particular, this means that we are expressly *not* thinking of manifolds with corners as stratified spaces with maps of stratified spaces as the only smooth maps. Indeed, our definition of smooth maps of manifolds with corners allow for the image of any open face of the source manifold with corners to be contained in multiple open faces of the target.

As explained in Appendix A of [13], de Rham cohomology is well-defined for manifolds with corners and is invariant under smooth homotopy. For a manifold with corners  $W$ , we denote by  $\mathring{W}$  the open dense interior of  $W$ . There always exists a manifold without corners  $\bar{W}$  into which  $W$  embeds; in this case, it is said that  $\bar{W}$  *contains*  $W$  as a domain.

Given two maps  $f : M \rightarrow N$  and  $g : M' \rightarrow N$ , the symbol  $M \times_N M'$  will denote the fiber product of  $M$  with  $M'$  over  $N$ . In the case where we wish to emphasize the maps  $f$  and  $g$ , we may write  $M \times_{f,N,g} M'$ .

For any topological space  $X$ ,  $\mathbf{Open}(X)$  will always denote the category of open subsets of  $X$  with morphisms inclusions of subsets. The symbols **Sets** and **Groupoids** will denote the categories of sets and (small) groupoids, respectively. By *presheaf of groupoids*, we will always mean a *strict* presheaf of groupoids with domain a (full subcategory of) the category of open subsets on some topological space; in other words, a (1-)functor  $\mathcal{F} : \mathbf{Open}(X)^{op} \rightarrow \mathbf{Groupoids}$ . To avoid unnecessary generality involving sites and categories fibered in groupoids, we will take *stack* to mean such a presheaf of groupoids satisfying some extra conditions (see Definition B.3).

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## 2. SYMPLECTIC TORIC MANIFOLDS

What follows is a review of the recently published classification of non-compact symplectic toric manifolds by Karshon and Lerman [13]. It is by no means a complete account; the aim is to give a rough outline of their classification. This section also serves as a convenient repository of relevant ideas we will later be citing and adapting. Those familiar with the result of Karshon and Lerman may safely skip this section.

Fix a torus  $G$  with Lie algebra  $\mathfrak{g}$ . Following Karshon and Lerman, we define unimodular local embeddings as follows.

**Definition 2.1** (Unimodular cones and unimodular local embeddings). A **unimodular cone** at  $\epsilon \in \mathfrak{g}^*$  is a subset of the form:

$$C = \{\eta \in \mathfrak{g}^* \mid \langle \eta - \epsilon, v_i \rangle \geq 0 \text{ for all } 1 \leq i \leq k\}$$

where  $\{v_1, \dots, v_k\}$  is the basis for an integral lattice of a subtorus of  $G$ . This cone is labeled  $C_{\{v_1, \dots, v_k\}, \epsilon}$ .

For a manifold with corners  $W$ , a smooth map  $\psi : W \rightarrow \mathfrak{g}^*$  is a **unimodular local embedding** if, for each  $w \in W$ , there exists neighborhood  $U$  of  $w$  in  $W$  so that  $\psi|_U$  is an open embedding

of  $U$  onto a neighborhood of the cone point of a unimodular cone  $C_w := C_{\{v_1, \dots, v_k\}, \psi(w)}$  for a valid tuple  $\{v_1, \dots, v_k\}$ .

**Remark 2.2.** Note that the unimodular cone  $C_{\{v_1, \dots, v_k\}, \epsilon}$  contains the affine subspace

$$A = \{\eta \in \mathfrak{g}^* \mid \langle \eta - \epsilon, v_i \rangle \text{ for } 1 \leq i \leq k\}$$

Furthermore, the subspace  $A - \epsilon$  is exactly the annihilator of the Lie algebra spanned by  $\{v_1, \dots, v_k\}$ .

Given a symplectic toric manifold  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  (i.e., a symplectic manifold  $(M, \omega)$  with Hamiltonian toric action of  $G$  satisfying  $2 \dim(G) = \dim(M)$  and moment map  $\mu : M \rightarrow \mathfrak{g}^*$ ), the orbital moment map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  is a unimodular local embedding.

**Proposition 2.3.** For a symplectic toric manifold  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$ , the quotient  $M/G$  is naturally a manifold with corners. Furthermore, for  $G$ -quotient map  $\pi : M \rightarrow M/G$ , the unique continuous map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  (called the *orbital moment map*) making the diagram

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow \mu & \\ M/G & \xrightarrow{\bar{\mu}} & \mathfrak{g}^* \end{array}$$

commute is a unimodular local embedding.

Given two symplectic toric manifolds  $(M, \omega, \mu)$  and  $(M', \omega', \mu')$ , if there is a  $G$ -equivariant symplectomorphism  $\varphi : (M, \omega) \rightarrow (M', \omega')$ , then it is clear that, for  $G$ -quotient map  $\pi : M' \rightarrow M'/G$ ,  $\pi \circ \varphi : M \rightarrow M'/G$  is a  $G$ -quotient map for  $M$ . If  $\varphi$  additionally preserves a chosen moment map for each, they must share an orbital moment map as well. Thus, to understand the collection of all symplectic toric manifolds, it makes sense to group symplectic toric manifolds together by quotient type and orbital moment map.

**Definition 2.4.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding. Then a **symplectic toric manifold over  $\psi$**  is a symplectic toric manifold  $(M, \omega, \mu)$  together with a  $G$ -quotient map  $\pi : M \rightarrow W$  such that  $\mu = \psi \circ \pi$ . This data will be expressed as the triple  $(M, \omega, \pi : M \rightarrow W)$ .

The groupoid of symplectic toric manifolds over  $\psi$ , denoted  $\text{STM}_\psi(W)$ , is the groupoid with

- objects: symplectic toric manifolds over  $\psi$ ; and
- morphisms:  $G$ -equivariant symplectomorphisms

$$f : (M, \omega, \pi : M \rightarrow W) \rightarrow (M', \omega', \pi' : M' \rightarrow W)$$

satisfying  $\pi' \circ f = \pi$ .

The strategy for actually classifying these spaces is to relate them to a simpler class of objects, namely symplectic toric bundles.

**Definition 2.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding. Then the **groupoid of symplectic toric principal  $G$ -bundles over  $\psi$** , denoted  $\text{STB}_\psi(W)$ , is the groupoid with

- objects: pairs  $(\pi : P \rightarrow W, \omega)$ , for  $\pi : P \rightarrow W$  a principal  $G$ -bundle and  $\omega$  a  $G$ -invariant symplectic form with moment map  $\psi \circ \pi$ ; and

- morphisms:  $G$ -equivariant symplectomorphisms

$$\varphi : (\pi : P \rightarrow W, \omega) \rightarrow (\pi' : P' \rightarrow W, \omega')$$

for which  $\pi' \circ \varphi = \pi$ .

**Remark 2.6.** For any open subset  $U$  of  $W$ ,  $\psi|_U : U \rightarrow \mathfrak{g}^*$  is also a unimodular local embedding; thus we may define

$$\mathbf{STB}_\psi(U) := \mathbf{STB}_{\psi|_U}(U) \quad \text{and} \quad \mathbf{STM}_\psi(U) := \mathbf{STM}_{\psi|_U}(U).$$

These collections of groupoids define presheaves of groupoids

$$\mathbf{STB}_\psi : \mathbf{Open}(W)^{op} \rightarrow \mathbf{Groupoids} \quad \text{and} \quad \mathbf{STM}_\psi : \mathbf{Open}(W)^{op} \rightarrow \mathbf{Groupoids},$$

where  $\mathbf{Open}(W)$  denotes the category of open subsets of  $W$  with inclusions. Indeed, for any pair of nested open subsets  $U \subset V$  of  $W$ , we have restriction functors  $\rho_{VU}$ , taking a symplectic toric manifold  $(M, \omega, \pi : M \rightarrow V)$  over  $\psi|_V : V \rightarrow \mathfrak{g}^*$  to the symplectic toric manifold over  $\psi|_U$

$$\rho_{VU}(M, \omega, \pi : M \rightarrow V) := (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}, \pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U).$$

And, since any morphism

$$f : (M, \omega, \pi : M \rightarrow V) \rightarrow (M', \omega', \pi' : M' \rightarrow V)$$

in  $\mathbf{STM}_\psi(V)$  must satisfy  $\pi' \circ f = \pi$ , it follows that  $f$  restricts to a well-defined morphism

$$\rho_{VU}(f) : \rho_{VU}(M, \omega, \pi : M \rightarrow V) \rightarrow \rho_{VU}(M', \omega', \pi' : M' \rightarrow V).$$

Finally, for  $U \subset U' \subset U''$ , it is easy to check that  $\rho_{U'U} \circ \rho_{U''U'} = \rho_{U''U}$ .

A similar argument confirms  $\mathbf{STB}_\psi$  is a presheaf as well. To avoid unnecessarily clunky notation,  $\rho_{VU}(M, \omega, \pi : M \rightarrow V)$  and  $\rho_{VU}(f)$  will be denoted as  $(M, \omega, \pi : M \rightarrow V)|_U$  and  $f|_U$ , respectively.

To establish the equivalence of the groupoids  $\mathbf{STB}_\psi(W)$  and  $\mathbf{STM}_\psi(W)$ , Karshon and Lerman introduce the functor  $c : \mathbf{STB}_\psi(W) \rightarrow \mathbf{STM}_\psi(W)$ , constructed with the following steps:

- (1) For every  $w \in W$ ,  $\psi$  identifies a basis  $\{v_1^{(w)}, \dots, v_k^{(w)}\}$  of a subtorus  $K_w$  of  $G$ . This is the basis of the subtorus identified by the unimodular cone onto which a neighborhood of  $w$  embeds. In turn, this basis defines a symplectic representation  $\rho : K_w \rightarrow Sp(V, \omega_V)$  on a symplectic vector space  $(V, \omega_V)$  (namely, the standard symplectic toric representation with weights  $\{v_1^*, \dots, v_k^*\}$ ).
- (2) For any principal bundle  $\pi : P \rightarrow W$ , let  $\sim$  be the equivalence relation

$$p \sim p' \text{ when there exists } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'$$

on  $P$ . Then define

$$c_{\text{Top}}(\pi : P \rightarrow W, \omega) := (P / \sim, \bar{\pi} : P / \sim \rightarrow W),$$

for  $\bar{\pi}$  the  $G$ -quotient of  $P / \sim$  descending from  $\pi$ . It follows from the  $G$ -equivariance of morphisms of  $\mathbf{STB}_\psi(W)$  that morphisms descend to the topological quotients modulo  $\sim$ . Thus, the relation establishes a functor

$$c_{\text{Top}} : \mathbf{STB}_\psi(W) \rightarrow \text{topological } G \text{ spaces over } W$$

This functor commutes with restrictions; that is, for every open  $U$  in  $W$ ,

$$c_{\text{Top}}((P, \omega)|_U) = c_{\text{Top}}(P, \omega)|_U := (\bar{\pi}^{-1}(U), \bar{\pi}).$$

- (3) To “symplectize” these topological quotients, Karshon and Lerman use symplectic cuts, showing for each  $w \in W$ , there is a neighborhood  $U_w$  of  $w$  in  $W$  (defined independent of  $P$ ) so that

$$\text{cut}((P, \omega)|_{U_w}) := (P|_{U_w} \times V_w) //_0 K_w$$

is a symplectic toric manifold over  $\psi|_{U_w}$ . This establishes a functor

$$\text{cut} : \text{STB}_\psi(U_w) \rightarrow \text{STM}_\psi(U_w)$$

for each  $w$ .

- (4) For each  $w$  and  $(P, \omega) \in \text{STB}_\psi(W)$ , there is a homeomorphism

$$\alpha_w^P : c_{\text{Top}}((P, \omega)|_{U_w}) \rightarrow \text{cut}((P, \omega)|_{U_w})$$

preserving the  $G$ -quotients of  $c_{\text{Top}}((P, \omega)|_{U_w})$  and  $\text{cut}((P, \omega)|_{U_w})$ . For any  $w, w'$  in  $W$  with  $U_w \cap U_{w'}$  non-empty,  $\alpha_{w'}^{P'} \circ (\alpha_w^P)^{-1}$  is a symplectomorphism. Therefore,  $c_{\text{Top}}((P, \omega))$  inherits the structure of a symplectic toric manifold.

- (5) Finally, for each isomorphism  $\varphi : (P, \omega) \rightarrow (P', \omega')$ , for any  $w \in W$ , the diagram

$$\begin{array}{ccc} c_{\text{Top}}(P, \omega)|_{U_w} & \xrightarrow{\alpha_w^P} & \text{cut}((P, \omega)|_{U_w}) \\ c_{\text{Top}}(\varphi) \downarrow & & \downarrow \text{cut}(\varphi|_{U_w}) \\ c_{\text{Top}}(P', \omega')|_{U_w} & \xrightarrow{\alpha_w^{P'}} & \text{cut}((P', \omega')|_{U_w}) \end{array}$$

commutes, and so the morphism  $c_{\text{Top}}(\varphi)$  is a symplectomorphism.

As it will be important later, we present below an outline of the process used to “symplectize” the quotient space  $c_{\text{Top}}(P, \omega)$ . First, an important theorem about extending Marsden-Weinstein and Meyer reduction to a specific scenario involving manifolds with corners is required.

**Theorem 2.7** (Theorem 2.23, [13]). Suppose  $(M, \sigma)$  is a symplectic manifold with corners with a proper Hamiltonian action of a Lie group  $K$  with moment map  $\Phi : M \rightarrow \mathfrak{k}^*$  (for  $\mathfrak{k}$  the Lie algebra of  $K$ ). Suppose also that:

- for each  $x \in \Phi^{-1}(0)$ , the stabilizer  $K_x$  of  $x$  is trivial;
- $\Phi$  admits an extension  $\tilde{\Phi}$  to a manifold  $\tilde{M}$  containing  $M$  as a domain; and
- $\tilde{\Phi}^{-1}(0) = \Phi^{-1}(0)$ .

Then  $\Phi^{-1}(0)$  is a manifold without corners and the reduction at 0

$$M //_0 K := \Phi^{-1}(0) / K$$

is naturally a symplectic manifold.

We now construct  $\text{cut}((P, \omega)|_{U_w})$  for a valid choice of  $U_w$ .

**Construction 2.8.** Fix a symplectic toric bundle  $(\pi : P \rightarrow W, \omega)$  over unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ .

Because  $\psi$  is a unimodular local embedding, there exists a unimodular cone  $C_{\{v_1, \dots, v_k\}, \psi(w)}$  so that  $\psi$  embeds a neighborhood of  $w$  in  $W$  onto a neighborhood of  $\psi(w)$  in the cone. Recall

this means  $\{v_1, \dots, v_k\}$  is the basis for the Lie algebra  $\mathfrak{k}$  of a subtorus  $K_w \subset G$ . In turn, these define a symplectic toric representation of  $K_w$ ,  $\rho_w : K_w \rightarrow (\mathbb{C}^k, \omega_{\mathbb{C}^k})$  with symplectic weights  $\{v_1^*, \dots, v_k^*\}$  (for  $\omega_{\mathbb{C}^k}$  the standard symplectic form on  $\mathbb{C}^k$ ); this has moment map

$$\mu_w : \mathbb{C}^k \rightarrow \mathfrak{k}^*, \quad (z_1, \dots, z_k) \mapsto - \sum_{i=1}^k |z_i|^2 v_i^*$$

Let  $\iota : \mathfrak{k} \rightarrow \mathfrak{g}$  be the embedding of  $\mathfrak{k}$  into  $\mathfrak{g}$  and let  $\iota^*$  be the dual to this embedding. Then, since  $\psi \circ \pi$  is the moment map for the free action of  $G$  on  $P$ ,  $\nu := \iota^* \circ \psi \circ \pi$  is the moment map for the action of  $K_w$  on  $P$ . Define  $\xi_0 := \iota^*(\psi(w))$ . Then, for  $C'_w$  the cone

$$C'_w := \{\xi \in \mathfrak{k}^* \mid \langle \xi - \xi_0, v_i \rangle \geq 0, 1 \leq i \leq k\},$$

we can identify the cone  $C_w$  with the product  $\mathfrak{k}^o \times C'_w$ . Here,  $\mathfrak{k}^o$  is the annihilator of  $\mathfrak{k}$  in  $\mathfrak{g}$  which is embedded in  $C_w$  as the affine space  $\mathfrak{k}^o + \psi(w)$ . This affine space corresponds to the open face of  $W$  containing  $w$  (near  $w$ ).

Thus, there exist contractible neighborhoods  $\mathcal{U}$  of  $w$  in the open face of  $W$  containing  $w$  and  $\mathcal{V}$  of  $\xi_0$  in  $\mathfrak{k}^*$  so that, for  $\mathcal{V}' := C'_w \cap \mathcal{V}$ , a neighborhood  $U_w$  is diffeomorphic to  $\mathcal{U} \times \mathcal{V}'$ . Let  $\nu : P|_{U_w} \rightarrow \mathcal{V}'$  be the map  $\iota^* \circ \psi \circ \pi$ . Then  $\nu$  is a trivializable  $\mathcal{U} \times G$  fiber bundle. Thus,  $P|_{U_w}$  is contained in a manifold  $\tilde{P}$  (diffeomorphic to  $\mathcal{V} \times \mathcal{U} \times G$ ) as a domain and  $\nu : P|_{U_w} \rightarrow \mathcal{V}'$  admits a smooth extension to a map  $\tilde{\nu} : \tilde{P} \rightarrow \mathcal{V}$ .

Define  $\Phi : P|_{U_w} \times V_w \rightarrow \mathfrak{k}^*$  by

$$\Phi(p, z) := \nu(p) - \iota^*(\psi(w)) + \mu_w(z)$$

Then  $\Phi$  is a moment map for the action of  $K$  on  $P|_{U_w} \times \mathbb{C}^k$  and admits an extension to the map

$$\tilde{\Phi}(p, z) := \tilde{\nu}(p) - \iota^*(\psi(w)) + \mu_w(z)$$

satisfying the conditions of Theorem 2.7. Thus, reduction at the zero level set of  $\Phi$  yields a symplectic manifold (without corners). One may check that  $(P|_U \times \mathbb{C}^k) //_0 K_w$  inherits a  $G$ -quotient map to  $U_w \bar{\pi}$  with respect to which  $((P|_U \times \mathbb{C}^k) //_0 K_w, \bar{\pi})$  is a symplectic toric manifold of  $\text{STM}_\psi(U_w)$ . Define  $\text{cut}((P, \omega)|_{U_w}) := ((P|_U \times \mathbb{C}^k) //_0 K_w, \bar{\pi})$ .

For  $\varphi : (P, \omega) \rightarrow (P', \omega')$ , the morphism  $\varphi \times id_{\mathbb{C}} : P|_{U_w} \times \mathbb{C}^k \rightarrow P'|_{U_w} \times \mathbb{C}^k$  descends to a symplectomorphism  $\text{cut}(\varphi) : \text{cut}((P, \omega)|_{U_w}) \rightarrow \text{cut}((P', \omega')|_{U_w})$ . ◇

For the purposes of this paper, it will also be important to sketch the construction of the homeomorphisms  $\alpha_w^P : c_{\text{Top}}(P, \omega)|_{U_w} \rightarrow \text{cut}((P, \omega)|_{U_w})$ .

**Construction 2.9.** For each  $w \in W$  and  $U_w$  defined as in Construction 2.8, to define the homeomorphisms  $\alpha_w^P : c_{\text{Top}}(P, \omega)|_{U_w} \rightarrow \text{cut}((P, \omega)|_{U_w})$ , first let  $s : \mu_w(\mathbb{C}^k) \rightarrow \mathbb{C}^k$  be the continuous section of  $\mu_w$  defined by

$$s(\eta) := \left( \sqrt{\langle -\eta, v_1 \rangle}, \dots, \sqrt{\langle -\eta, v_k \rangle} \right)$$

Then one can show that the map  $\alpha_w^P : c_{\text{Top}}(P|_{U_w}) \rightarrow (P|_{U_w} \times \mathbb{C}^k) //_0 K_w$  defined by

$$[p] \mapsto [p, s(\iota^*(\psi(p)) - \nu(p))]$$

is a well-defined  $G$ -equivariant homeomorphism. ◇



**Remark 2.10.** For  $w \in \overset{\circ}{W}$  (the interior of  $W$ ), we have that  $\psi|_{U_w}$  is an open embedding into  $\mathfrak{g}^*$  itself (i.e., rather than just an embedding into a cone). This means that  $K_w$  is trivial and therefore  $\text{cut}((P, \omega)|_{U_w}) = (P|_{U_w}, \omega, \pi)$  as symplectic toric manifolds over  $\psi$ .

**Remark 2.11.** Since the collection of functors  $c : \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$  for each open  $U$  in  $W$  commute with restriction, it follows that we have a map of presheaves

$$c : \text{STB}_\psi \rightarrow \text{STM}_\psi.$$

In service of classifying the groupoid of symplectic toric bundles over a given unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , Karshon and Lerman prove the following lemmas (which we restate as they will become important later in this paper).

**Lemma 2.12** (Lemma 3.2, [13]). Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding, let  $\pi : P \rightarrow W$  be a principal  $G$ -bundle, and let  $A \in \Omega^1(P, \mathfrak{g})^G$  be a connection 1-form for  $P$ . For convenience, define  $\mu := \psi \circ \pi$ . Then:

- Any closed  $G$ -invariant 2-form on  $P$  with moment map  $\mu$  is automatically symplectic; this includes the form  $d\langle \mu, A \rangle$ .
- The map from closed 2-forms on  $W$  to closed 2-forms on  $P$ :

$$\beta \mapsto d\langle \mu, A \rangle + \pi^* \beta$$

establishes a bijection between the set of closed 2-forms on  $W$  and the set of  $G$ -invariant symplectic forms on  $P$  with moment map  $\mu$ .

This has an obvious corollary that will be important for us later (though was not explicitly mentioned by Karshon and Lerman).

**Corollary 2.13.** For  $\psi$  and  $P$  as in the lemma above, let  $\omega$  be any closed  $G$ -invariant 2-form on  $P$  with moment map  $\mu$ . Then the map from closed 2-forms on  $W$  to closed 2-forms on  $P$ :

$$\beta \mapsto \omega + \pi^* \beta$$

also establishes a bijection between closed 2-forms on  $W$  and  $G$ -invariant symplectic forms on  $P$  with moment map  $\mu$ .

**Lemma 2.14** (Lemma 3.3, [13]). Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding and let  $\pi : P \rightarrow W$  be a principal  $G$ -bundle. For any 1-form  $\gamma$  on  $W$  and any  $G$ -invariant symplectic form  $\omega$  on  $P$  with moment map  $\mu$ , there exists a gauge transformation  $f : P \rightarrow P$  with  $f^*(\omega + \pi^*(d\gamma)) = \omega$ .

Karshon and Lerman then go on to prove that, for every open subset  $U$  of  $W$ ,  $c_U : \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$  is a fully faithful functor. Observing that, for contractible open subsets  $V$  of  $W$ , the groupoid  $\text{STM}_\psi$  is connected (i.e., all objects are isomorphic), they also conclude that  $c$  must be locally essentially surjective. Implicitly using the fact that  $\text{STB}_\psi$  is a stack and  $\text{STM}_\psi$  is a prestack (see Appendix B), they are able to conclude the following theorem.

**Theorem 2.15** (Theorem 4.1, [13]). Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding. Then

$$c : \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$$

is an equivalence of categories.

Using the tools of Lemmas 2.12 and Lemma 2.14, Karshon and Lerman are able to show that the elements of  $\mathbf{STB}_\psi(W)$  are classified by the cohomology classes  $H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$ , (where  $\mathbb{Z}_G := \ker(\exp : \mathfrak{g} \rightarrow G)$  is the integral lattice). Thus, using the equivalence of categories  $c$ , they conclude the following result.

**Theorem 2.16** (Theorem 1.3, [13]). Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding. Then:

- (1) The groupoid  $\mathbf{STM}_\psi(W)$  is non-empty; that is, there exists symplectic toric manifold  $(M, \omega, \mu)$  with  $G$ -quotient  $\pi : M \rightarrow W$  with respect to which  $\psi$  is the orbital moment map.
- (2)  $\pi_0(\mathbf{STM}_\psi(W))$ , the set of isomorphism classes of  $\mathbf{STM}_\psi(W)$ , is in bijective correspondence with the cohomology classes:

$$H^2(W, \mathbb{Z}_G \times \mathbb{R}) \cong H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$$

Since  $c$  is in fact an isomorphism of presheaves and, it can be shown the identification of symplectic toric bundles with elements of  $H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$  commutes with restrictions as well, it is fitting to call the elements of  $H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$  characteristic classes for symplectic toric manifolds over  $\psi$ .

## Part I: Classifying symplectic toric cones

As symplectic toric stratified spaces are built from symplectic toric cones, to understand the former spaces, it is necessary to understand the latter. In Section 3, we fully describe these cones as well as their orbital moment maps. Recall that a symplectic toric manifold  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  is a *symplectic toric cone* if  $M$  has a free and proper action of  $\mathbb{R}$  commuting with the action of  $G$  and, with respect to any action diffeomorphism  $\rho_\lambda : M \rightarrow M$  for this  $\mathbb{R}$  action (for  $\lambda \in \mathbb{R}$ ), we have  $\rho_\lambda^* \omega = e^\lambda \omega$ . Additionally, we impose that the moment map  $\mu$  for  $M$  is the *homogeneous moment map* for  $(M, \omega)$ , satisfying  $\mu(t \cdot p) = e^t \mu(p)$  for every  $t \in \mathbb{R}$  and  $p \in M$  (such a moment map for  $(M, \omega)$  always exists).

Since any symplectic toric cone  $(M, \omega, \mu)$  is, in particular, a symplectic toric manifold, it follows, as in [13], that the  $G$ -quotient  $M/G$  is a manifold with corners and the orbital moment map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  is a type of map known as a unimodular local embedding. As a consequence of  $\mu$  being homogeneous, we may conclude that  $\bar{\mu}$  satisfies two additional properties: the quotient  $M/G$  inherits a free and proper  $\mathbb{R}$  action and, with respect to this action,  $\bar{\mu}$  is itself homogeneous. Given an arbitrary manifold with corners  $W$  for which there is a unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , we call  $\psi$  a *homogeneous unimodular local embedding* if it and  $W$  satisfies these additional properties.

As in the case of symplectic toric manifolds, it makes sense to group together symplectic toric cones by orbital moment map: for any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , we define *the groupoid of symplectic toric cones over  $\psi$* , denoted  $\mathbf{STC}_\psi(W)$ , as the groupoid with objects symplectic toric cones admitting a  $G$ -quotient map to  $W$  for which  $\psi$  is the orbital moment map and with morphisms symplectomorphisms preserving these quotients that are both  $G$  and  $\mathbb{R}$ -equivariant. It is important to note that we may not initially be sure this groupoid is non-empty.

For any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  and for any  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ ,  $\psi|_U$  is a homogeneous unimodular local embedding as well. It follows

that, for  $\text{Open}_{\mathbb{R}}(W)$  the category of  $\mathbb{R}$ -invariant open subsets of  $W$ , we may form a presheaf of groupoids

$$\text{STC}_{\psi} : \text{Open}_{\mathbb{R}}(W)^{op} \rightarrow \text{Groupoids}$$

In Section 4, we define *homogeneous symplectic toric bundles over  $\psi$*  for any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ . These are pairs  $(\pi : P \rightarrow W, \omega)$  for  $\pi : P \rightarrow W$  a principal  $G$ -bundle over the manifold with corners  $W$  and  $\omega$  a  $G$ -invariant symplectic form on  $P$  with moment map  $\psi \circ \pi$ . Additionally,  $P$  comes with an  $\mathbb{R}$  action making  $(P, \omega, \psi \circ \pi)$  a symplectic toric cone. Taking a map of homogeneous symplectic toric bundles over  $\psi$  to be any isomorphism of principal  $G$ -bundles over  $W$  that is both a symplectomorphism and  $\mathbb{R}$ -equivariant, we may then define *the groupoid of homogeneous symplectic toric bundles over  $\psi$* , denoted  $\text{HSTB}_{\psi}(W)$ .

As in the case of symplectic toric cones, homogeneous symplectic toric bundles also define a presheaf

$$\text{HSTB}_{\psi} : \text{Open}_{\mathbb{R}}(W)^{op} \rightarrow \text{Groupoids}$$

We also describe in this section some of the important properties of homogeneous symplectic toric bundles. Of particular note is Proposition 4.6, in which we show that every principal  $G$ -bundle  $\pi : P \rightarrow W$  with an  $\mathbb{R}$ -action for which  $\pi$  is  $\mathbb{R}$ -equivariant admits a  $G$ -invariant symplectic form  $\omega$  with respect to which  $(P, \omega)$  is a homogeneous symplectic toric bundle. In Proposition 4.10, we show that any two homogeneous symplectic toric bundles over the same homogeneous unimodular local embedding are isomorphic exactly when they have the same structure as a principal  $G$ -bundle with free  $\mathbb{R}$  action.

In Section 5, we define the map of presheaves  $\text{hc} : \text{HSTB}_{\psi} \rightarrow \text{STC}_{\psi}$ . In essence, this is a version of the map  $c$  of Karshon and Lerman taking symplectic toric bundles to symplectic toric manifolds that remembers the  $\mathbb{R}$  action of a homogeneous symplectic toric bundle. In showing that the category  $\text{HSTB}_{\psi}(W)$  is non-empty for any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , this functor allows us to conclude that the groupoid  $\text{STC}_{\psi}(W)$  must be non-empty as well. In Theorem 5.7, we show that  $\text{hc}$  is an isomorphism of presheaves over  $\text{Open}_{\mathbb{R}}(W)$ . With this in mind, we may focus on identifying the isomorphism classes of homogeneous symplectic toric bundles.

In Section 6, we provide characteristic classes for symplectic toric cones. This is done via Proposition 6.5, which shows that, for every homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , the isomorphism classes of  $\text{STC}_{\psi}(W)$  are in bijective correspondence with the isomorphism classes of  $BG(W)$  (the groupoid of principal  $G$ -bundles over  $W$ ). This allows us to conclude that these bundles admit characteristic classes of the form  $H^2(W, \mathbb{Z}_G)$ , for  $\mathbb{Z}_G$  the integral lattice  $\ker(\exp : \mathfrak{g} \rightarrow G) \subset \mathfrak{g}$  (this is the content of Proposition 6.8). Finally, we are able to use the isomorphism of presheaves  $\text{hc}$  to conclude Theorem 1.2: the isomorphism classes of  $\text{STC}_{\psi}(W)$  are in bijective correspondence with the cohomology classes  $H^2(W, \mathbb{Z}_G)$ .

In Section 7, we take a quick detour from classifying symplectic toric stratified spaces to give characteristic classes for contact toric manifolds. We exploit the intimate relationship between symplectic toric cones and contact toric manifolds (described in Theorem 7.3) to relate symplectic toric cones with contact toric manifolds. This allows us to finish this part of the paper by proving Theorem 1.3, classifying the contact toric manifolds with symplectizations admitting the structure of a symplectic toric manifold over a homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  with the cohomology classes  $H^2(W, \mathbb{Z}_G)$ .

### 3. SYMPLECTIC TORIC CONES

In this section, we review symplectic toric cones and describe their orbital moment maps; we call these maps homogeneous unimodular local embeddings. We also define the category of symplectic toric cones over a choice of such an orbital moment map.

As above, fix a torus  $G$  with Lie algebra  $\mathfrak{g}$ .

**Definition 3.1.** A **symplectic cone** is any symplectic manifold  $(M, \omega)$  together with a free and proper  $\mathbb{R}$  action satisfying  $\rho_\lambda^* \omega = e^\lambda \omega$  for each real number  $\lambda \in \mathbb{R}$  with action diffeomorphism  $\rho_\lambda : M \rightarrow M$ .

For  $(M, \omega)$  a symplectic cone with an action of  $G$ , we call a triple

$$(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$$

a **symplectic toric cone** if

- the actions of  $G$  and  $\mathbb{R}$  on  $M$  commute;
- the action of  $G$  on  $(M, \omega)$  is a symplectic toric action with moment map  $\mu$ ; and
- $\mu : M \rightarrow \mathfrak{g}^*$  is the **homogeneous moment map** for  $(M, \omega)$ : for every  $\lambda \in \mathbb{R}$  and  $p \in M$ ,  $\mu(\lambda \cdot p) = e^\lambda \mu(p)$ .

As choices of moment maps for  $(M, \omega)$  differ only by constants, it follows that this homogeneous moment map is unique. The existence of such a moment map is well-known.

Basic known information and properties of symplectic cones and their relationship with contact manifolds has been relegated to Appendix A. We now describe the form any orbital moment map to a symplectic toric cone must take.

**Definition 3.2.** Given a manifold with corners  $W$  with a free and proper action of  $\mathbb{R}$ , a unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  (see Definition 2.1) is a **homogeneous unimodular local embedding** if  $\psi(t \cdot w) = e^t \psi(w)$  for every  $t \in \mathbb{R}$  and  $w \in W$ .

There is more simply verified set of conditions a unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  can satisfy in order to be a homogeneous unimodular local embedding. Furthermore, there is at most one action of  $\mathbb{R}$  on  $W$  with respect to which  $\psi$  is homogeneous.

**Proposition 3.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a unimodular local embedding. Suppose also that the image  $\psi(W)$  doesn't contain  $0 \in \mathfrak{g}^*$  and is closed under the radial action of  $\mathbb{R}$  on  $\mathfrak{g}^*$  (i.e., for every  $w \in W$  and  $t \in \mathbb{R}$ ,  $e^t \psi(w) \in \psi(W)$ ). Then there exists a unique free  $\mathbb{R}$  action on  $W$  satisfying  $\psi(t \cdot w) = e^t \psi(w)$ . Furthermore, this action is proper, and therefore  $\psi$  is a homogeneous unimodular local embedding.

*Proof.* Let  $R$  be the radial vector field on  $\mathfrak{g}^*$ . As  $\psi(W)$  is closed under radial scaling,  $R$  restricts to a complete vector field on the manifold with corners  $\psi(W)$ . Then, as  $\psi$  is a local embedding, there exists a vector field  $\Xi$  with  $\psi_* \Xi$ . On each open subset  $U$  of  $W$  for which  $\psi|_U$  is an open embedding to a unimodular cone, the integral curves of  $\Xi$  correspond to the integral curves of  $R$  and so, since  $R$  is complete, we may arbitrarily extend the integral curves of  $\Xi$ . Thus,  $\Xi$  is complete, and so the flow of  $\Xi$  induces a well-defined smooth action of  $\mathbb{R}$  on  $W$ .

Again, as the integral curves of  $\Xi$  push forward to the integral curves of  $R$ , it follows that  $\psi(t \cdot w) = e^t \psi(w)$  (as  $v \mapsto e^t v$  is the time  $t$  flow of  $R$ ). As the image of  $\psi$  lands in  $\mathfrak{g}^* \setminus \{0\}$ , it follows also that the action of  $\mathbb{R}$  on  $W$  must be free.

To see this is the unique  $\mathbb{R}$  action on  $W$  with this property, fix an element  $w \in W$  and a real number  $t \in \mathbb{R}$ . Let  $I$  be the straight line connecting  $\psi(w)$  and  $e^t\psi(w)$  in  $\mathfrak{g}^*$ . Then, for an action of  $\mathbb{R}$  on  $W$  with the properties prescribed above, let  $\gamma$  be the curve  $\gamma : [0, t] \rightarrow W$  defined by  $\gamma(s) := s \cdot w$ . It follows that the image of  $\psi \circ \gamma$  is  $I$ . Using a finite cover of the image of  $\gamma$  by open sets in  $W$  on which  $\psi$  is an open embedding, we may conclude that, for each  $s \in [0, t]$ ,  $\gamma(s)$  is the unique value  $s \cdot w$  can take.

To see this action is proper, fix a compact subset  $C \subset W \times W$ . Then for  $C_i$  the projection of  $C$  onto the  $i^{\text{th}}$  factor of  $W \times W$ , let  $D = C_1 \cup C_2$ . Then  $D \times D$  is a compact subset of  $W \times W$  containing  $C$ . Note that, since  $\psi(D)$  doesn't contain 0 and is compact, it is contained in a bounded annulus; i.e.,  $\psi(D)$  is bounded and there is an open ball  $B$  around 0 in  $\mathfrak{g}^*$  with  $B \cap \psi(D) = \emptyset$ .

Now, pick a cover of  $D$  in  $W \times W$  by sets of the form  $U \times V$ , where the restriction of  $\psi$  to each  $U$  and  $V$  is an open embedding onto  $\psi(W)$ ,  $\psi(U)$  and  $\psi(V)$  are bounded, and so that the closure of  $\psi(U)$  and  $\psi(V)$  in  $\mathfrak{g}^*$  do not contain 0 (that is, so that  $\psi(U)$  and  $\psi(V)$  lie in bounded annuli of  $\mathfrak{g}^*$  as well). Let  $\{U_i \times V_i\}_{i=1}^n$  be a finite subcover of this cover.

For each  $U_i \times V_i$ , define the number  $a_i$ :

$$a_i = \sup\{t \geq 0 \mid \text{there exists } (u, v) \in U_i \times V_i \text{ such that } t \cdot v = u \text{ or } (-t) \cdot v = u\}$$

By our choice of  $U_i \times V_i$ , each  $a_i$  exists. Let  $A$  be the maximum of the  $a_i$  for  $1 \leq i \leq n$ .

Finally, for

$$\begin{aligned} \Phi : \mathbb{R} \times W &\rightarrow W \times W \\ (t, w) &\mapsto (t \cdot w, w) \end{aligned}$$

the action map, it follows that  $\Phi^{-1}(C)$  is compact: it is a closed set and is contained inside of the set  $[-A, A] \times D \subset \mathbb{R} \times W$  which is itself compact.  $\square$

**Proposition 3.4.** Let  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  be a symplectic toric cone. Then for any  $G$ -quotient  $\pi : M \rightarrow M/G$  of  $M$ , the orbital moment map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  is a homogeneous unimodular local embedding.

*Proof.* We have already that  $M/G$  is a manifold with corners and that  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  is a unimodular local embedding (see Proposition 2.3). Since  $\mu$  is homogeneous, it follows that  $\bar{\mu}(W)$  is closed under the radial action of  $\mathbb{R}$  on  $\mathfrak{g}^*$ . Finally, note that the image of  $\mu$  does not contain zero (see Proposition A.11). Then by Proposition 3.3, there is a free and proper  $\mathbb{R}$  action on  $M/G$  with respect to which  $\bar{\mu}$  is a homogeneous unimodular local embedding.  $\square$

We now group symplectic toric cones together by orbital moment map.

**Definition 3.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then a *symplectic toric cone over  $\psi$*  is a symplectic toric cone  $(M, \omega, \mu)$  together with a  $G$ -quotient  $\pi : M \rightarrow W$  so that  $\mu = \psi \circ \pi$ . This data is represented by the triple  $(M, \omega, \pi : M \rightarrow W)$ .

Denote by  $\text{STC}_\psi(W)$  the groupoid of symplectic toric cones over  $\psi$ , the groupoid with

- objects: symplectic toric cones over  $\psi$ ; and
- morphisms:  $(G \times \mathbb{R})$ -equivariant symplectomorphisms

$$\varphi(M, \omega, \pi : M \rightarrow W) \rightarrow (M', \omega', \pi' : M' \rightarrow W)$$

satisfying  $\pi' \circ \varphi = \pi$ .

Before continuing, we now prove another consequence of Proposition 3.3 that will be very useful later in the paper. It's proof first requires the following lemma.

**Lemma 3.6.** Let  $H$  and  $K$  be Lie groups and let  $K$  be compact. Let  $X$  is a Hausdorff topological space on which  $H$  and  $K$  have commuting actions and let  $\pi : X \rightarrow X/K$  be the quotient. Then the action of  $H$  descends to a continuous action on  $X/K$  and, if this action is proper, then the action of  $H$  on  $X$  is proper as well.

*Proof.* That the action of  $H$  descends to an action on  $X/K$  is simply a consequence of the fact that the actions of  $H$  and  $K$  on  $X$  commute. Then we have the following commutative diagram:

$$\begin{array}{ccc} H \times X & \xrightarrow{\Phi} & X \times X \\ (id, \pi) \downarrow & & \downarrow (\pi, \pi) \\ H \times X/K & \xrightarrow{\bar{\Phi}} & X/K \times X/K \end{array}$$

where  $\Phi$  and  $\bar{\Phi}$  are the action maps  $\Phi(h, x) := (h \cdot x, x)$  and  $\bar{\Phi}(h, \cdot[x]) := (h \cdot [x], [x])$ .

Let  $C$  be a compact subset of  $X \times X$ . We then have that  $(\pi, \pi)(C)$  is a compact subset of  $X/K \times X/K$ . Then, since we assume the action of  $H$  on  $X/K$  is proper,  $\bar{\Phi}^{-1}(\pi(C))$  is a compact subspace of  $H \times X/K$ . Since  $\pi$  is proper (as shown in Theorem 3.1, pp. 38 of [4]), it follows that  $(id, \pi)^{-1}(\bar{\Phi}^{-1}(\pi(C)))$  is a compact subspace of  $H \times X$ .

Finally, note that by the commutativity of the above diagram,  $\Phi^{-1}(C) \subset (id, \pi)^{-1}(\bar{\Phi}^{-1}(\pi(C)))$ . Then since  $X \times X$  is Hausdorff,  $C$  is closed and therefore  $\Phi^{-1}(C)$  is closed as well. As a closed subset of a compact set, we may conclude that  $\Phi^{-1}(C)$  is compact in Hausdorff  $H \times X$ .  $\square$

The previous two results now have an easy but important corollary.

**Proposition 3.7.** Let  $(M, \omega, \mu)$  be a symplectic toric manifold. Suppose further that  $M$  has a free  $\mathbb{R}$  action commuting with the action of  $G$  such that

- the orbital moment map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  is a homogeneous unimodular local embedding (with respect to the  $\mathbb{R}$  action descending from  $M$  to  $M/G$ ); and
- For each  $\lambda \in \mathbb{R}$  with action diffeomorphism  $\rho_\lambda : M \rightarrow M$ ,  $\rho_\lambda^* \omega = e^\lambda \omega$ .

Then  $(M, \omega, \mu)$  is a symplectic toric cone. In other words, given all of the other ingredients for a symplectic toric cone, the properness of the free  $\mathbb{R}$  action comes for free.

*Proof.* Since  $\bar{\mu}$  is a homogeneous unimodular local embedding with respect to the action of  $\mathbb{R}$  on  $M/G$ , Proposition 3.3 tells us that this action is proper. Since this is the action descending from  $M$ , it follows that the quotient map  $\pi : M \rightarrow M/G$  is  $\mathbb{R}$ -equivariant. Therefore, by Lemma 3.6, the  $\mathbb{R}$  action on  $M$  is proper and so  $(M, \omega, \mu)$  is a symplectic toric cone.  $\square$

We now build slices for the  $\mathbb{R}$  action on  $W$  for any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ . This allows us to conclude that any  $\mathbb{R}$ -quotient map  $q : W \rightarrow W/\mathbb{R}$  is a principal  $\mathbb{R}$ -bundle of manifolds with corners. These slices built in a particular way for use in the construction of the functor  $\text{hc}$  (see Section 5).

**Lemma 3.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. For any point  $w \in W$ , recall from Definition 2.1 that there exists a unimodular cone  $C = C_{\{v_1, \dots, v_k\}, \psi(w)}$  so that  $\psi$  embeds a neighborhood of  $w$  in  $W$  onto a neighborhood of  $\psi(w)$  in  $C$ . Let  $\mathfrak{k}$  the Lie algebra associated to the subtorus  $K$  determined by  $\{v_1, \dots, v_k\}$  and let  $C'$  be the

unimodular cone  $C_{\{v_1, \dots, v_k\}, 0} \subset \mathfrak{k}^*$ . Then there is an open  $\mathbb{R}$ -invariant neighborhood  $U_w$  of  $w$  such that

- (1)  $\psi|_{U_w} : U_w \rightarrow \mathfrak{g}^*$  is an open embedding onto a neighborhood of  $\psi(w)$  in  $C$ ; and
- (2) there is a contractible open subset  $\mathcal{U}$  of the sphere  $S^{\dim(G)-k-1}$  and a contractible open neighborhood  $\mathcal{V}$  of 0 in  $\mathfrak{k}^*$  so that  $\psi(U_w)$  is  $\mathbb{R}$ -equivariantly diffeomorphic to  $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C')$  (where  $\psi(U_w)$  inherits the radial action of  $\mathbb{R}$  on  $\mathfrak{k}^*$  and we extend the action of  $\mathbb{R}$  on itself trivially to the product  $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C')$ )

*Proof.* Let  $\iota : \mathfrak{k} \rightarrow \mathfrak{g}$  be the inclusion and  $\iota^* : \mathfrak{k}^* \rightarrow \mathfrak{g}^*$  the dual to this inclusion. We first show that  $\iota^*(\psi(w)) = 0$ . As noted in Remark 2.2, the unimodular cone  $C = C_{\{v_1, \dots, v_k\}, \psi(w)}$  contains the affine subspace

$$A = \{\eta \in \mathfrak{g}^* \mid \langle \eta - \psi(w), v_i \rangle = 0\}.$$

Since  $\psi$  is homogeneous, the image of  $\psi$  contains the ray  $\{t\psi(w) \mid t > 0\}$ . In particular, this means that  $A$ , as an affine subspace of  $\mathfrak{k}$ , must contain the origin. It follows  $A$  is an honest linear subspace and, since  $\psi(w) \in A$ ,  $A - \psi(w) = A$ . As  $A - \psi(w) = \mathfrak{k}^o$ , we may then conclude that  $\psi(w)$  is in  $\mathfrak{k}^o$ . Thus,  $\iota^*(\psi(w)) = 0$ .

By choosing a section of  $\iota^*$  embedding  $\mathfrak{k}^*$  into  $\mathfrak{g}^*$ , we have an identification of vector spaces  $\mathfrak{k}^* \times \mathfrak{k}^o \cong \mathfrak{g}^*$  with  $\psi(w)$  corresponding to the point  $(0, \psi(w))$  in  $\mathfrak{k}^* \times \mathfrak{k}^o$ . This identification descends to an identification  $C \cong C' \times \mathfrak{k}^o$  that respects scalar multiplication when defined. Since  $C'$  is a unimodular cone based at the origin,  $C'$  is closed under the radial action of  $\mathbb{R}$  on  $\mathfrak{k}^*$  and therefore it follows that  $C$  is closed under the radial action of  $\mathbb{R}$  on  $\mathfrak{g}^*$ . Thus, the identification  $C \cong C' \times \mathfrak{k}^o$  is  $\mathbb{R}$ -equivariant.

Now, note that, for  $S(\mathfrak{k}^o)$  the sphere for the vector space  $\mathfrak{k}^o$  defined with respect to some fixed norm, there is an  $\mathbb{R}$ -equivariant identification  $\mathfrak{k}^o \setminus \{0\} \cong \mathbb{R} \times S(\mathfrak{k}^o)$ . Therefore, we have an  $\mathbb{R}$ -equivariant diffeomorphism

$$f : C \setminus \{0\} \rightarrow \mathbb{R} \times S(\mathfrak{k}^o) \times C'$$

for which  $f(\psi(w)) = (\lambda, x, 0)$ , for some pair  $(\lambda, x) \in \mathbb{R} \times S(\mathfrak{k}^o)$ .

So let  $U$  be a neighborhood of  $w$  in  $W$  so that  $\psi|_U : U \rightarrow \mathfrak{g}^*$  is an open embedding onto a neighborhood of the cone point  $\psi(w)$  of  $C$ . Then  $U$  contains a subset  $\Sigma$  containing  $w$  so that

$$f(\psi(\Sigma)) = \{\lambda\} \times \mathcal{U} \times (\mathcal{V} \cap C')$$

for  $\mathcal{U}$  a contractible open subset of  $S(\mathfrak{k}^o)$  and  $\mathcal{V}$  a contractible open subset of 0 in  $\mathfrak{k}^o$ . As  $\psi$  and  $f$  are both  $\mathbb{R}$ -equivariant, each  $\mathbb{R}$ -orbit of  $W$  intersects  $\Sigma$  at most once.

Define  $U_w := \mathbb{R} \cdot \Sigma$ . Since  $\psi$  is  $\mathbb{R}$ -equivariant and is injective on  $\Sigma$  and since the action of  $\mathbb{R}$  on  $\psi(W)$  is free, we may conclude that  $\psi$  is injective on  $U_w$ . Therefore, since  $\psi$  is locally an open embedding, it follows that  $\psi$  yields an open embedding of  $U_w$ . As it is more or less clear that

$$f(\psi(U_w)) = \mathbb{R} \cdot (\{\lambda\} \times \mathcal{U} \times (\mathcal{V} \cap C'))$$

is open in  $\mathbb{R} \times S(\mathfrak{k}^o) \times C'$ , we may conclude that  $U_w$  is an open neighborhood of  $w$ .

To finish, note that, as the action of  $\mathbb{R}$  on  $\mathcal{V} \cap C'$  is just radial scaling,

$$\mathbb{R} \cdot (\{\lambda\} \times \mathcal{U} \times (\mathcal{V} \cap C')) \cong \mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C')$$

where we are trivially extending the action of  $\mathbb{R}$  on itself to the product  $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C')$ .  $\square$

We may now conclude the following result.

**Proposition 3.9.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $q : W \rightarrow W/\mathbb{R}$  be an  $\mathbb{R}$ -quotient map for the free  $\mathbb{R}$  action on  $W$ . Then  $q$  is a principal  $\mathbb{R}$ -bundle in the category of manifolds with corners.

*Proof.* From Lemma 3.8, it follows that each point  $w \in W$  has an  $\mathbb{R}$ -invariant neighborhood  $U_w$  equivariantly diffeomorphic to  $\mathbb{R} \times V$ , for  $V$  an open subset of  $[0, \infty)^k \times \mathbb{R}^{\dim(G)-k}$ . Thus,  $[w] \in W/\mathbb{R}$  has a neighborhood homeomorphic to  $V$ . These neighborhoods in the quotient are clearly coherent and give a (possibly non-Hausdorff) manifold with corners structure on  $W/\mathbb{R}$ .

Since  $\psi|_{U_w}$  is an open embedding, and since  $(\mathfrak{g}^* \setminus \{0\})/\mathbb{R} \cong S^{\dim(G)-1}$  is Hausdorff,  $\psi(v)$  and  $\psi(v')$  are separable by  $\mathbb{R}$ -invariant neighborhoods. Thus,  $W/\mathbb{R}$  is Hausdorff.

Finally, note that our slices naturally give us smooth local trivializations of  $W$  as a principal  $\mathbb{R}$ -bundle.  $\square$

Like symplectic toric manifolds over a specific unimodular local embedding, symplectic toric cones over a homogeneous unimodular local embedding also form a presheaf. Rather than using a site of open subsets over a topological subspace, we instead consider a smaller site.

**Definition 3.10.** Let  $W$  be a manifold with corners with a free  $\mathbb{R}$  action. Then let  $\mathbf{Open}_{\mathbb{R}}(W)$  be the full subcategory of  $\mathbf{Open}(W)$  of  $\mathbb{R}$ -invariant subsets of  $W$ . That is, the category with objects  $\mathbb{R}$ -invariant open subset of  $W$  and morphisms inclusions of open subsets.

**Proposition 3.11.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then

$$U \mapsto \mathbf{STC}_{\psi}(U) := \mathbf{STC}_{\psi|_U}(U)$$

defines a presheaf over  $\mathbf{Open}_{\mathbb{R}}(W)$ .

*Proof.* For each  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ ,  $\psi|_U$  is still a homogeneous unimodular local embedding. Thus, the groupoid  $\mathbf{STC}_{\psi}(U) := \mathbf{STC}_{\psi|_U}(U)$  is well-defined. For  $U \subset V$   $\mathbb{R}$ -invariant open subsets of  $W$ , we define restriction by

$$(M, \omega, \pi : M \rightarrow V)|_U := (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}, \pi|_{\pi^{-1}(U)})$$

To see this is still a symplectic cone, note that, since the actions of  $G$  and  $\mathbb{R}$  on  $M$  commute, the action of  $\mathbb{R}$  on  $M$  descends to an action on  $V$  with respect to which  $\pi$  is  $\mathbb{R}$ -equivariant. It follows from Proposition 3.3 that this action matches the action on  $V$  with respect to which  $\psi|_V$  is homogeneous (as both actions satisfy the hypotheses of the aforementioned proposition). Therefore, since  $U$  is  $\mathbb{R}$ -invariant, the set  $\pi^{-1}(U)$  is  $\mathbb{R}$ -invariant as well.

Because morphisms of  $\mathbf{STC}_{\psi}(V)$  must cover the identity on  $V$ , any morphism restricts to a morphism over  $U$  in  $\mathbf{STC}_{\psi}(U)$ . It is easy to check that, with these restriction morphisms,

$$\mathbf{STC}_{\psi} : \mathbf{Open}_{\mathbb{R}}(W)^{op} \rightarrow \mathbf{Groupoids}, \quad U \mapsto \mathbf{STC}_{\psi}(U)$$

is a presheaf.  $\square$

#### 4. HOMOGENEOUS SYMPLECTIC TORIC BUNDLES

To classify symplectic toric cones, we use homogeneous symplectic toric bundles together with an analogue of the isomorphism of presheaves  $c : \mathbf{STB}_{\psi} \rightarrow \mathbf{STM}_{\psi}$  from [13] (see Section 2). In this section, we present a definition and some properties of homogeneous symplectic toric bundles.



**Definition 4.1.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then a homogeneous symplectic toric bundle over  $\psi$  is a symplectic toric bundle  $(\pi : P \rightarrow W, \omega)$  (see Definition 2.5) together with a free and proper  $\mathbb{R}$  action so that

- The actions of  $G$  and  $\mathbb{R}$  on  $P$  commute;
- $(P, \omega)$  is a symplectic cone with respect to the given  $\mathbb{R}$  action; and
- $\psi \circ \pi$  is a *homogeneous* moment map for the action of  $G$  on  $(P, \omega)$ .

These bundles are represented by pairs  $(\pi : P \rightarrow W, \omega)$ .

Denote by  $\text{HSTB}_\psi(W)$  the groupoid of homogeneous symplectic toric bundles over  $W$ . This is the groupoid with objects homogeneous symplectic toric bundles  $(\pi : P \rightarrow W, \omega)$  and morphisms  $\varphi : (\pi : P \rightarrow W, \omega) \rightarrow (\pi' : P' \rightarrow W, \omega')$   $\mathbb{R}$ -equivariant gauge transformations such that  $\varphi^*\omega' = \omega$ .

While choosing  $\psi \circ \pi$  to be a homogeneous moment map is the “correct” condition to impose from the standpoint of creating a coherent definition, there is a simpler and more useful condition we now provide.

**Lemma 4.2.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  is a homogeneous unimodular local embedding. Suppose that  $(\pi : P \rightarrow W, \omega)$  is a symplectic toric bundle with a free and proper  $\mathbb{R}$  action, commuting with the action of  $G$ , with respect to which  $(P, \omega)$  is a symplectic cone. Then  $(\pi : P \rightarrow W, \omega)$  is a homogeneous symplectic toric bundle if and only if  $\pi$  is  $\mathbb{R}$ -equivariant.

*Proof.* Since  $(\pi : P \rightarrow W, \omega)$  is a symplectic toric bundle,  $\psi \circ \pi$  is a moment map for the action of  $G$  on  $(P, \omega)$ . Thus, all that remains to be shown is that  $\psi \circ \pi$  is homogeneous if and only if  $\pi$  is  $\mathbb{R}$ -equivariant.

If  $\pi$  is  $\mathbb{R}$ -equivariant, then for any  $p \in P$ ,  $\psi(\pi(t \cdot p)) = \psi(t \cdot (\pi(p))) = e^t \psi(\pi(p))$ , so  $\psi \circ \pi$  is homogeneous. On the other hand, since the actions of  $\mathbb{R}$  and  $G$  on  $P$  commute, the free action of  $\mathbb{R}$  on  $P$  descends to a free action on  $W$  with respect to which  $\pi$  is  $\mathbb{R}$ -equivariant and with respect to which  $\psi(t \cdot w) = e^t \psi(w)$ . By Proposition 3.3, this implies that this induced  $\mathbb{R}$  action matches the  $\mathbb{R}$  action on  $W$  with respect to which  $\psi$  is a homogeneous unimodular local embedding.  $\square$

As in the case of symplectic toric cones, the collection of groupoids of homogenous symplectic toric bundles over restrictions of  $\psi$  to  $\mathbb{R}$ -invariant open subsets of  $W$  is a presheaf of groupoids over  $\text{Open}_\mathbb{R}(W)$ .

**Proposition 4.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then the function on  $\text{Open}_\mathbb{R}(W)$

$$U \mapsto \text{HSTB}_\psi(U) := \text{HSTB}_{\psi|_U}((U))$$

together with the appropriate restriction morphisms defines a presheaf of groupoids.

As the justification here is more or less the same as that for Proposition 3.11, we omit the proof.

**Remark 4.4.** In fact,  $\text{HSTB}_\psi : \text{Open}_\mathbb{R}(W)^{op} \rightarrow \text{Groupoids}$  is a stack. This fact will be important later, but as the proof is essentially a marginally adjusted retelling of the proof that the presheaf of principal bundles over a site is a stack, we relegate the proof to the appendix (Proposition B.8).

As in the case of symplectic toric bundles, it is not immediately clear that the category  $\text{HSTB}_\psi(W)$  is non-empty. However, a  $G$ -invariant symplectic form  $\omega$  for any principal  $G$ -bundle  $P$  over  $W$  with appropriate  $\mathbb{R}$  action can be built from a choice of connection 1-form on  $P$ . Before showing this, we need a technical lemma.

**Lemma 4.5.** Let  $\pi : P \rightarrow B$  be a principal  $G$ -bundle of manifolds with corners. Further, suppose  $P$  and  $B$  admit free actions of  $\mathbb{R}$  with respect to which  $\pi$  is  $\mathbb{R}$ -equivariant and the  $\mathbb{R}$ -quotient  $q' : B \rightarrow B/\mathbb{R}$  is a principal  $\mathbb{R}$ -bundle of manifolds with corners. Let  $q : P \rightarrow P/\mathbb{R}$  be an  $\mathbb{R}$ -quotient. Finally, suppose that the actions of  $G$  and  $\mathbb{R}$  on  $P$  commute.

Then  $P/\mathbb{R}$  admits the structure of a manifold with corners. Furthermore, there is a smooth map  $\varpi : P/\mathbb{R} \rightarrow B/\mathbb{R}$  such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & P/\mathbb{R} \\ \pi \downarrow & & \downarrow \varpi \\ B & \xrightarrow{q'} & B/\mathbb{R} \end{array} \quad (4.1)$$

commutes. Finally, the maps  $\varpi : P/\mathbb{R} \rightarrow B/\mathbb{R}$  and  $q : P \rightarrow P/\mathbb{R}$  are a principal  $G$ -bundle and principal  $\mathbb{R}$ -bundle of manifolds with corners, respectively.

*Proof.* We will first work only topologically. The existence of  $\varpi$  is a consequence of the universal property of a quotient: as  $\pi$  is  $\mathbb{R}$ -equivariant and  $q'$  is  $\mathbb{R}$ -invariant, the composition  $q' \circ \pi$  must collapse  $\mathbb{R}$ -orbits. Thus, there exists a unique map  $\varpi : P/\mathbb{R} \rightarrow B/\mathbb{R}$  making diagram (4.1) commute.

Since the actions of  $G$  and  $\mathbb{R}$  commute on  $P$ , the free action of  $G$  on  $P$  descends to a free action of  $G$  on  $P/\mathbb{R}$ .

Let  $U$  be any contractible open subset of  $B/\mathbb{R}$  and let  $s : U \rightarrow B$  be a local section of the principal  $\mathbb{R}$ -bundle  $q' : B \rightarrow B/\mathbb{R}$ . This induces an  $\mathbb{R}$ -equivariant homeomorphism  $\varphi_s : B|_U \rightarrow U \times \mathbb{R}$  with  $\varphi_s^{-1}(b, t) := t \cdot s(b)$ . It follows that  $B|_U$  is contractible, so we may find another local section  $s' : B|_U \rightarrow P$  of the principal  $G$ -bundle  $\pi : P \rightarrow B$ .

Now, define  $\varphi_s^i$  to be the homeomorphism  $\varphi_s$  followed by the projection onto the  $i^{\text{th}}$  factor of the product  $U \times \mathbb{R}$ . We may adjust  $s'$  to an  $\mathbb{R}$ -equivariant section  $t$  by defining:

$$t : B|_U \rightarrow P \quad b \mapsto \varphi_s^2(b) \cdot s'(\varphi_s^1(b))$$

This yields a  $(G \times \mathbb{R})$ -equivariant homeomorphism  $\varphi_t : P|_{B|_U} \rightarrow B|_U \times G$  with  $\varphi_t^{-1}(b, g) = g \cdot t(b)$ .

Using the notation  $\varphi_t^i$  as above, we have a  $(G \times \mathbb{R})$ -equivariant homeomorphism

$$\phi : P|_{B|_U} \rightarrow U \times \mathbb{R} \times G, \quad p \mapsto (\varphi_s(\varphi_t^1(p)), \varphi_t^2(p))$$

Since we have that

$$\varpi \circ q \circ t \circ s = q' \circ \pi \circ t \circ s = \text{id}_{B/\mathbb{R}}$$

$q \circ t \circ s$  is a section of  $\varpi$  and therefore  $q(P|_{B|_U}) = P/\mathbb{R}|_U$ . Thus,  $\phi$  descends to a  $G$ -equivariant homeomorphism

$$\bar{\phi} : P/\mathbb{R}|_U \rightarrow U \times G$$

and we may conclude  $\varpi$  is a (topological) principal  $G$ -bundle.

Now, note that we may choose smooth sections  $s$  and  $t$  above. Then  $\varphi_s$  and  $\varphi_t$  must both be diffeomorphisms and so the homeomorphisms  $\bar{\phi}$  as above defined for each contractible

subset  $U$  of  $B/\mathbb{R}$  define a smooth manifold with corners structure on  $P/\mathbb{R}$ . It is clear then that, with this smooth structure in mind,  $\varpi$  is smooth and  $P/\mathbb{R}$  has smooth trivializations as a principal  $G$ -bundle; therefore,  $\varpi : P/\mathbb{R} \rightarrow B/\mathbb{R}$  is a principal  $G$ -bundle of manifolds with corners.  $\square$

Now, we build a symplectic form for any principal  $G$ -bundle  $\pi : P \rightarrow W$  with an appropriate  $\mathbb{R}$  action.

**Proposition 4.6.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $\pi : P \rightarrow W$  be any principal  $G$ -bundle with a free action of  $\mathbb{R}$  commuting with the action of  $G$  such that  $\pi$  is  $\mathbb{R}$ -equivariant. Then there exists a connection 1-form  $A \in \Omega^1(P, \mathfrak{g}^*)^G$  so that  $(\pi : P \rightarrow W, d\langle A, \psi \circ \pi \rangle)$  is a homogeneous symplectic toric bundle with respect to this  $\mathbb{R}$  action.

*Proof.* First, note that it is proven in [13] that, for any connection 1-form  $A$ ,  $d\langle \psi \circ \pi, A \rangle$  is a  $G$ -invariant symplectic form for  $P$  with respect to which  $\psi \circ \pi$  is a moment map (see Lemma 2.12). So it remains to show we can find a particular connection satisfying the additional conditions required of a homogeneous symplectic toric bundle.

Let  $Q := P/\mathbb{R}$  and  $B := W/\mathbb{R}$  with  $\mathbb{R}$ -quotient maps  $q' : W \rightarrow B$  and  $q : P \rightarrow Q$ . From Lemma 4.5, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{q} & Q \\ \pi \downarrow & & \downarrow \varpi \\ W & \xrightarrow{q'} & B \end{array}$$

where  $\varpi : Q \rightarrow B$  is a principal  $G$ -bundle and  $q : P \rightarrow Q$  is a principal  $\mathbb{R}$ -bundle.

So, the bundle  $q : P \rightarrow Q$  is trivializable: there is a gauge transformation  $\phi : P \rightarrow Q \times \mathbb{R}$  of principal  $\mathbb{R}$ -bundles over  $Q$ . As  $q$  is  $G$ -equivariant, it follows that, with respect to the  $G$  action on  $Q$  extended trivially to  $Q \times \mathbb{R}$ ,  $\phi$  is  $G$ -equivariant. Thus,  $\pi \circ \phi^{-1} : Q \times \mathbb{R} \rightarrow W$  is a principal  $G$ -bundle. On the other hand,  $\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \rightarrow B \times \mathbb{R}$  is of course also a principal  $G$ -bundle. As connection 1-forms on principal bundles must only satisfy conditions related to the associated  $G$  action, we may define a connection 1-form for the bundle  $\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \rightarrow B \times \mathbb{R}$  which will also be a connection for the bundle  $\pi \circ \phi^{-1} : Q \times \mathbb{R} \rightarrow W$ .

Let  $A'$  be any connection 1-form on  $\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \rightarrow B \times \mathbb{R}$  extended trivially from a connection 1-form on  $\varpi : Q \rightarrow B$ . Define  $A := \phi^* A'$ . We must show that  $d\langle \psi \circ \pi, A \rangle$  satisfies the necessary conditions for a symplectic form of a symplectic cone. Fix a real number  $\lambda$  and let  $\rho_\lambda$  be the diffeomorphism associated to its action on  $P$ . Let  $\rho'_\lambda$  be the diffeomorphism associated to the action of  $\lambda$  on  $Q \times \mathbb{R}$ . Then, as  $\phi$  is  $\mathbb{R}$ -equivariant,  $\phi \circ \rho_\lambda = \rho'_\lambda \circ \phi$ . As  $A'$  came from a connection on  $Q$ , it follows that  $\rho'^*_\lambda A' = A'$ . Using these facts, we calculate:

$$\begin{aligned} \rho_t^* d\langle \psi \circ \pi, A \rangle &= d\langle \psi \circ \pi \circ \rho_\lambda, \rho_\lambda^*(\phi^* A') \rangle \\ &= d\langle e^\lambda \cdot (\psi \circ \pi), \phi^*(\rho'^*_\lambda A') \rangle \\ &= d\langle e^\lambda \langle \psi \circ \pi, \phi^* A' \rangle \rangle \\ &= e^\lambda d\langle \psi \circ \pi, A \rangle \end{aligned}$$

This is exactly the condition  $d\langle\psi\circ\pi, A\rangle$  must satisfy so that  $(P, d\langle\psi\circ\pi, A\rangle)$  is a symplectic cone. Thus, using Proposition 3.7, we may conclude that the action of  $\mathbb{R}$  on  $P$  is proper and that  $(\pi : P \rightarrow W, d\langle\psi\circ\pi, A\rangle)$  is a homogenous symplectic toric bundle over  $\psi$ .  $\square$

We will soon show that two homogeneous symplectic toric bundles are isomorphic in  $\text{HSTB}_\psi$  exactly when there is an  $\mathbb{R}$ -equivariant gauge transformation between them. To prove this, we need the following lemma.

**Lemma 4.7.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $\pi : P \rightarrow W$  be a principal  $G$ -bundle with a free  $\mathbb{R}$  action commuting with the action of  $G$  so that  $\pi$  is  $\mathbb{R}$ -equivariant. Suppose  $\omega$  and  $\omega'$  are two symplectic forms so that  $(\pi : P \rightarrow W, \omega)$  and  $(\pi : P \rightarrow W, \omega')$  are both homogeneous symplectic toric bundles. Then the form  $\omega - \omega'$  is basic and, for  $\omega - \omega' = \pi^*\beta$ ,  $\beta$  is exact. Furthermore, there is a primitive  $\gamma$  of  $\beta$  satisfying:

$$\rho_\lambda^*\gamma = e^\lambda\gamma$$

for any  $\lambda \in \mathbb{R}$  with action diffeomorphism  $\rho_\lambda : W \rightarrow W$ .

*Proof.* First note that  $(\pi : P \rightarrow W, \omega)$  and  $(\pi : P \rightarrow W, \omega')$  are, in particular, symplectic toric bundles. Fix a connection 1-form  $A$  for which  $(\pi : P \rightarrow W, d\langle A, \psi \rangle)$  is a homogeneous symplectic toric bundle (as constructed in Proposition 4.6). Then  $\omega - d\langle A, \psi \rangle$  and  $\omega' - d\langle A, \psi \rangle$  are both basic (see Lemma 2.12); thus, the difference  $\omega - \omega'$  is basic as well.

Fix a real number  $\lambda$ . Writing  $\tau_\lambda : P \rightarrow P$  for the action isomorphism of  $\lambda$  on  $P$ , we have by assumption that  $\tau_\lambda^*\omega = e^\lambda\omega$  and  $\tau_\lambda^*\omega' = e^\lambda\omega'$ . So, of course, their difference  $\pi^*\beta$  must satisfy this condition as well.

As  $\pi$  is  $\mathbb{R}$ -equivariant, we have that  $\pi \circ \tau_\lambda = \rho_\lambda \circ \pi$ . So, we calculate:

$$\begin{aligned} \pi^*(\rho_\lambda^*\beta) &= \tau_\lambda^*(\pi^*\beta) \\ &= e^\lambda\pi^*\beta \\ &= \pi^*(e^\lambda\beta) \end{aligned}$$

Since  $\pi$  is a submersion, it follows that  $\rho_\lambda^*\beta = e^\lambda\beta$ .

Finally, write  $\Xi$  for the vector field on  $W$  with flow the action of  $\mathbb{R}$ . Then  $\beta$  satisfies  $L_\Xi\beta = \beta$  meaning, since  $\beta$  is closed, that  $\gamma := \iota_\Xi\beta$  is a primitive for  $\beta$ . It is easy to show that  $\gamma$  satisfies  $L_\Xi\gamma = \gamma$  as well. It thereby follows that  $\gamma$  satisfies the conditions hypothesized above.  $\square$

With this lemma in mind, we may prove the following important lemma (adapted from a lemma of [13]).

**Lemma 4.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $(\pi : P \rightarrow W, \omega)$  be a homogeneous symplectic toric bundle. Let  $\gamma$  be a 1-form on  $W$  satisfying  $\rho_\lambda^*\gamma = e^\lambda\gamma$  for every real  $\lambda$  with action diffeomorphism  $\rho_\lambda : W \rightarrow W$ . Then there is an isomorphism of homogeneous symplectic toric bundles  $\varphi : (\pi : P \rightarrow W, \omega) \rightarrow (\pi : P \rightarrow W, \omega + \pi^*d\gamma)$ .

**Remark 4.9.** It is clear that, for  $\gamma$  as in the lemma above, the proof of Lemma 4.7 may be reversed to conclude that  $(\pi : P \rightarrow W, \omega + \pi^*d\gamma)$  is indeed a homogeneous symplectic toric bundle over  $\psi$ .

*Proof.* We will essentially repeat the proof of Lemma 3.3 of [13] with the addition of an  $\mathbb{R}$  action; for the convenience of the reader, we will sketch the borrowed details.

To build the map  $f$ , we may use Moser's deformation method on the family of symplectic forms:

$$\omega_t = \omega + t\pi^*d\gamma, \quad t \in [0, 1].$$

Then there is a unique time-dependent vector field  $X_t$  on  $P$  satisfying:

$$\iota_{X_t}\omega_t = -\pi^*\gamma. \quad (4.2)$$

By showing that  $X_t$  is  $G$ -invariant and tangent to the compact fibers of  $\pi$ , we may conclude that the time 1 flow of  $X_t$  exists and  $G$ -equivariant. Therefore, for  $\varphi : P \rightarrow P$  this time 1 flow, we must have  $\pi \circ \varphi = \pi$  and, as is standard in the use of Moser's method (as in [22]),  $\varphi$  satisfies

$$\varphi^*(\omega + \pi^*d\gamma) = \varphi^*(\omega_1) = \varphi^*(\omega_0) = \varphi^*(\omega).$$

It remains to be shown for our case that this gauge transformation is  $\mathbb{R}$ -equivariant. It is enough to show that the time dependent vector field  $X_t$  determined by the family of symplectic forms above is  $\mathbb{R}$ -invariant. Fix a real number  $\lambda$  and let  $\rho_\lambda : P \rightarrow P$  be the action diffeomorphism for  $\lambda$ . It is clear, as  $\rho_\lambda^*\omega = e^\lambda\omega$  and  $\rho_\lambda^*(\pi^*\gamma) = e^\lambda\pi^*\gamma$ , that  $\omega_t$  must satisfy the analogous property.

We calculate:

$$\iota_{(\rho_\lambda)_*X_t}\omega_t = \rho_{-\lambda}^*(\iota_{X_t}(\rho_\lambda^*\omega_t)) = \rho_{-\lambda}^*(\iota_{X_t}e^\lambda\omega_t) = e^\lambda\rho_{-\lambda}^*(-\pi^*\gamma) = e^\lambda e^{-\lambda}(-\pi^*\gamma) = -\pi^*\gamma$$

Because equation (4.2) uniquely determines the vector field  $X_t$ , it follows that  $(\rho_\lambda)_*X_t = X_t$ . Thus,  $X_t$  is  $\mathbb{R}$ -invariant, meaning its time 1 flow  $\varphi$  must be  $\mathbb{R}$ -equivariant.  $\square$

From the previous two lemmas, we may easily conclude the following proposition:

**Proposition 4.10.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then two homogeneous symplectic toric bundles over  $\psi$  ( $\pi : P \rightarrow W, \omega$ ) and ( $\pi' : P' \rightarrow W, \omega'$ ) are isomorphic as elements of  $\text{STC}_\psi$  if and only if there exists an  $\mathbb{R}$ -equivariant isomorphism of principal  $G$ -bundles  $\varphi : P \rightarrow P'$ .

*Proof.* Because an isomorphism in  $\text{HSTB}_\psi$  is in particular an  $\mathbb{R}$ -equivariant isomorphism of principal  $G$ -bundles, one direction is given by definition.

So suppose there exists an  $\mathbb{R}$ -equivariant isomorphism of principal  $G$ -bundles  $\varphi : P \rightarrow P'$ . Then, by Lemma 4.7, it follows that the difference  $\varphi^*(\omega') - \omega$  is basic. Furthermore, writing  $\varphi^*(\omega') - \omega = \pi^*\beta$ , there is a primitive  $\gamma$  of  $\beta$  satisfying  $\rho_\lambda^*\gamma = e^\lambda\gamma$ , where  $\rho_\lambda : W \rightarrow W$  is the action diffeomorphism for the action of real number  $\mathbb{R}$ . Then, by Lemma 4.8, we have that there exists an  $\mathbb{R}$ -equivariant gauge transformation  $\phi : P \rightarrow P$  satisfying  $\phi^*(\omega + d\pi^*\gamma) = \omega$ . Therefore,  $\varphi \circ \phi$  is an isomorphism of homogeneous symplectic toric bundles.  $\square$

## 5. THE MORPHISM OF PRESHEAVES $\text{hc} : \text{HSTB}_\psi \rightarrow \text{STC}_\psi$

In this section, we introduce a functor  $\text{hc} : \text{HSTB}_\psi(W) \rightarrow \text{STC}_\psi(W)$ . We then show that  $\text{hc}$  is an equivalence of categories; in fact, thinking of  $\text{HSTB}_\psi$  and  $\text{STC}_\psi$  as presheaves over  $\text{Open}_\mathbb{R}(W)$ ,  $\text{hc}$  is an isomorphism of presheaves. This functor is essentially a homogeneous version of the equivalence of categories  $c : \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$  of [13] (see Section 2). We first must verify that, given a homogeneous symplectic toric bundle  $(\pi : P \rightarrow W, \omega)$  over

homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , the  $\mathbb{R}$  action on  $P$  descends to an  $\mathbb{R}$  action on  $c(P, \omega)$  making this into a symplectic cone.

**Proposition 5.1.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $(\pi : P \rightarrow W, \omega)$  be a homogeneous symplectic toric bundle over  $\psi$ . Then, regarding  $(P, \omega)$  simply as a symplectic toric bundle over  $\psi$ , the symplectic toric manifold  $c(P, \omega)$  inherits an  $\mathbb{R}$  action from  $(P, \omega)$  with respect to which  $c(P, \omega)$  is a symplectic toric cone over  $\psi$ .

*Proof.* First, recall that  $c(P, \omega)$  is built first as a topological  $G$ -space  $c_{\text{Top}}(P, \omega) := P / \sim$ , where  $\sim$  is the equivalence relation:

$$p \sim p' \text{ when there exists } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'.$$

This topological quotient is then “symplectized” through the process detailed in Construction 2.8.

Now, as the actions of  $G$  and  $\mathbb{R}$  on  $P$  commute, it follows that the action of  $\mathbb{R}$  descends to a continuous action on  $c_{\text{Top}}(P, \omega)$ . To confirm the action is smooth, we will show that we may carefully repeat the cuts giving symplectic structure to  $c(P, \omega)$ , using  $\mathbb{R}$ -invariant subsets  $U \subset W$  to symplectize  $c_{\text{Top}}(P, \omega)$  via the reductions  $(P|_U \times \mathbb{C}^k) / {}_0K_w$ .

Fix an element  $w \in W$ . Recall that, since  $\psi$  is a unimodular local embedding, there is a neighborhood of  $w$  diffeomorphic via  $\psi$  to a neighborhood of  $\psi(w)$  in the unimodular cone  $C_w := C_{\{v_1, \dots, v_k\}, \psi(w)}$ , where  $\{v_1, \dots, v_k\}$  is a basis for the Lie algebra  $\mathfrak{k}$  for a subtorus  $K_w \leq G$ . As before, let  $\iota : \mathfrak{k} \rightarrow \mathfrak{g}$  be the natural inclusion with dual  $\iota^* : \mathfrak{g}^* \rightarrow \mathfrak{k}^*$ . As in Construction 2.8, define the cone  $C'_w$  by

$$C'_w := \{\xi \in \mathfrak{k}^* \mid \langle \xi, v_i \rangle \geq 0, 1 \leq i \leq k\}.$$

Then by Lemma 3.8, we may find an  $\mathbb{R}$ -invariant neighborhood  $U_w$  of  $w$ , a contractible open subset  $\mathcal{U}$  of the sphere  $S^{\dim(G)-k-1}$ , and a contractible open subset  $\mathcal{V}$  of the origin in  $\mathfrak{k}^*$  so that  $\psi|_{U_w}$  is an open embedding and  $U_w$  is diffeomorphic to  $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C'_w)$ . It is easy to confirm that, with respect to this identification,  $(\iota^* \circ \psi)(t, u, v) = e^t v$ . Since  $U_w$  is contractible,  $\pi : P|_{U_w} \rightarrow U_w$  is a trivializable principal  $G$ -bundle. So, since the map

$$\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C'_w) \times G \rightarrow \mathfrak{k}^* \quad (t, u, v, g) \mapsto e^t v$$

admits an extension

$$\mathbb{R} \times \mathcal{U} \times \mathcal{V} \times G \rightarrow \mathfrak{k}^* \quad (t, u, v, g) \mapsto e^t v,$$

there is a manifold  $\tilde{P}$  (isomorphic to  $\mathbb{R} \times \mathcal{U} \times \mathcal{V} \times G$ ) containing  $P|_{U_w}$  as a domain so that the map  $\nu := \iota^* \circ \psi \circ \pi : P|_{U_w} \rightarrow \mathfrak{k}^*$  admits an extension to  $\tilde{\nu} : \tilde{P} \rightarrow \mathfrak{k}^*$ .

Now, let  $K_w \rightarrow (\mathbb{C}^k, \omega_{\mathbb{C}^k})$  be the symplectic representation with weights  $\{v_1^*, \dots, v_k^*\}$  (here,  $\omega_{\mathbb{C}^k}$  denotes the standard symplectic form on  $\mathbb{C}^k$ ). We fix the moment map

$$\mu_w : \mathbb{C}^k \rightarrow \mathfrak{k}^* \quad \mu_w((z_1, \dots, z_k)) := - \sum_{j=1}^k |z_j|^2 v_j$$

for this spspace. Then the  $K_w$  action on  $(P|_{U_w} \times \mathbb{C}^k, \omega \oplus \omega_{\mathbb{C}^k})$  has moment map  $\Phi(p, z) := \nu(p) + \mu_w(z)$  and this clearly has extension  $\tilde{\Phi}(p, z) := \tilde{\nu}(p) + \mu_w(z)$  to the domain  $\tilde{P} \times \mathbb{C}^k$  containing  $P|_{U_w} \times \mathbb{C}^k$ .

The condition  $(p, z) \in \tilde{\Phi}^{-1}(0)$  imposes that  $\tilde{\nu}(p) = -\mu_w(z)$ , meaning the image of  $\tilde{\nu}$  must be contained in  $C'_w$ . It therefore follows that  $\Phi^{-1}(0) = \tilde{\Phi}^{-1}(0)$ . Thus, by Theorem 2.7, the reduction  $(P|_{U_w} \times \mathbb{C}^k) / {}_0K_w$  is a symplectic manifold.

As we've proceeded using (essentially) the same method as in Construction 2.8, it follows we may use the same form of homeomorphisms (as defined in Construction 2.9) to symplectize  $c_{\text{Top}}(P, \omega)$ . To finish, we need only show that there are compatible smooth  $\mathbb{R}$  actions on each  $(P|_{U_w} \times \mathbb{C}^k) //_0 K_w$  with respect to which the inherited symplectic form on  $c(P, \omega)$  is homogeneous.

So let  $\mathbb{R}$  act on  $P|_{U_w} \times \mathbb{C}^k$  via the  $\mathbb{R}$  action on  $P$  restricted to  $P|_{U_w}$  and via the “half-radial action” on  $\mathbb{C}^k$ : the action  $t \cdot z := e^{\frac{1}{2}t}z$ .  $\mu_w : \mathbb{C}^k \rightarrow \mathfrak{k}^*$  is homogeneous with respect to this action of  $\mathbb{R}$  on  $\mathbb{C}^k$  and, as  $\nu : P|_{U_w} \rightarrow \mathfrak{k}^*$  is homogeneous as well, it follows that the action of  $\mathbb{R}$  preserves the level set  $\Phi^{-1}(0)$ . Since the actions of  $K_w$  and  $\mathbb{R}$  commute, the action of  $\mathbb{R}$  descends to a smooth action on  $(P|_{U_w} \times \mathbb{C}^k) //_0 K_w$ .

It is easy to show that the transition homeomorphisms

$$\alpha_w^P : c_{\text{Top}}(P|_{U_w}) \rightarrow (P|_{U_w} \times \mathbb{C}^k) //_0 K_w$$

(again, as outlined in Construction 2.9) are  $\mathbb{R}$ -equivariant, where  $c_{\text{Top}}(P|_{U_w})$  and  $(P|_{U_w} \times \mathbb{C}^k) //_0 K_w$  inherit the  $\mathbb{R}$  actions described above. Thus, the action of  $\mathbb{R}$  on  $c_{\text{Top}}(P)$  inherited by the commutativity of the action of  $G$  and  $\mathbb{R}$  on  $P$  is in fact a smooth action on the symplectic manifold  $c(P, \omega)$ .

Finally, to see that the symplectic form  $\eta$  on  $c(P, \omega)$  is homogeneous (that is, satisfies  $\rho_\lambda^* \eta = e^\lambda \eta$  for the action diffeomorphism  $\rho_\lambda$  defined for each  $\lambda \in \mathbb{R}$ ), recall that, on the open dense interior  $\mathring{W}$  of  $W$ , the functor  $c$  is the identity (see Remark 2.10). In other words, for an open subset  $U \subset \mathring{W}$ ,  $(P|_U, \omega, \pi : P|_U \rightarrow U) = c(P, \omega)|_U$  as symplectic toric manifolds over  $\psi|_U$ . Thus,

$$\rho_\lambda^*(\eta|_U) = \rho_\lambda^*(\omega|_U) = e^\lambda \omega|_U = e^\lambda \eta|_U$$

As this identity holds on the open dense subset  $c(P, \omega)|_{\mathring{W}}$  of  $c(P, \omega)$ , it follows it must hold over all  $c(P, \omega)$ . Therefore, the above action of  $\mathbb{R}$  on  $c(P, \omega)$  renders  $c(P, \omega)$  a symplectic toric cone (the properness of the  $\mathbb{R}$  action on  $M$  is ensured by Proposition 3.7).  $\square$

**Definition 5.2.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then define  $\text{hc} : \text{HSTB}_\psi(W) \rightarrow \text{STC}_\psi(W)$  to be the functor taking a homogeneous symplectic toric bundle  $(P, \omega)$  to the symplectic manifold  $c(P, \omega)$  with  $\mathbb{R}$  action inherited from  $(P, \omega)$ , as outlined in Proposition 5.1. For a morphism  $\varphi : (P, \omega) \rightarrow (P', \omega')$ , we may take  $\text{hc}(\varphi) := c(\varphi)$ . It is easy to check that, since  $\varphi$  is  $(G \times \mathbb{R})$ -equivariant,  $\text{hc}(\varphi)$  is  $(G \times \mathbb{R})$ -equivariant as well. It is also easy to confirm that, as with  $c$ ,  $\text{hc} : \text{HSTB}_\psi \rightarrow \text{STC}_\psi$  is a map of presheaves.

To prove that  $\text{hc} : \text{HSTB}_\psi \rightarrow \text{STC}_\psi$  is an isomorphism of presheaves, we first must prove that  $\text{hc}_U$  is a fully faithful functor for each  $\mathbb{R}$ -invariant open subset  $U \subset W$ . We use the following lemma.

**Lemma 5.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then the forgetful functors  $\iota_h : \text{HSTB}_\psi(W) \rightarrow \text{STB}_\psi(W)$  and  $\iota_c : \text{STC}_\psi(W) \rightarrow \text{STM}_\psi(W)$  are faithful.

*Proof.*  $\iota_h$  takes a homogeneous symplectic toric bundle to the underlying symplectic toric bundle and  $\iota_c$  takes a symplectic toric cone to the underlying symplectic toric manifold (in other words, both functors “forget” the  $\mathbb{R}$  action on the respective source objects). The morphisms in both source categories are just the morphisms of the target category that happen to be  $\mathbb{R}$ -equivariant.

It is more or less obvious that, as forgetful functors,  $\iota_h$  and  $\iota_c$  are faithful.  $\square$

Now, we show that  $\mathbf{hc}$  is fully faithful.

**Lemma 5.4.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then for every  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ , the functor  $\mathbf{hc}_U : \mathbf{HSTB}_\psi(U) \rightarrow \mathbf{STC}_\psi(U)$  is fully faithful.

*Proof.* Note that, as  $\psi|_U : U \rightarrow \mathfrak{g}^*$  is also a homogeneous unimodular local embedding and the groupoid  $\mathbf{HSTB}_\psi(U)$  is, by definition, the groupoid  $\mathbf{HSTB}_{\psi|_U}(U)$ , we need only worry about the case of  $U = W$  as this will generalize to any  $\mathbb{R}$ -invariant open subset  $U \subset W$ .

Consider the following diagram

$$\begin{array}{ccc} \mathbf{HSTB}_\psi(W) & \xrightarrow{\iota_h} & \mathbf{STB}_\psi(W) \\ \mathbf{hc} \downarrow & & \downarrow c \\ \mathbf{STC}_\psi(W) & \xrightarrow{\iota_c} & \mathbf{STM}_\psi(W) \end{array}$$

where  $\iota_h$  and  $\iota_c$  are the faithful functors of Lemma 5.3. From the definition of each functor, it follows quite easily that this diagram commutes.

As  $c \circ \iota_h = \iota_c \circ \mathbf{hc}$ , it follows that  $\iota_c \circ \iota_h$  is faithful (as both  $c$  and  $\iota_h$  are faithful). Therefore, since  $\iota_c$  is faithful, it follows that  $\mathbf{hc}$  must be faithful.

To finish, we need to show  $\mathbf{hc}$  is full. Fix two homogeneous symplectic toric bundles  $(P, \omega)$  and  $(P', \omega')$  in  $\mathbf{HSTB}_\psi(W)$ . Let

$$f : \mathbf{hc}(\pi : P \rightarrow W, \omega) \rightarrow \mathbf{hc}(\pi' : P' \rightarrow W, \omega')$$

be a map of symplectic toric cones. Applying  $\iota_c$ , we get an  $\mathbb{R}$ -equivariant map of symplectic toric manifolds

$$\iota_c(f) : \iota_c(\mathbf{hc}(P, \omega)) \rightarrow \iota_c(\mathbf{hc}(P', \omega'))$$

which, by the commutativity of the above diagram, is in fact a map

$$\iota_c(f) : c(\iota_h(P, \omega)) \rightarrow c(\iota_h(P', \omega')).$$

As  $c$  is full, there exists a map of symplectic toric bundles

$$\varphi : \iota_h(P, \omega) \rightarrow \iota_h(P', \omega')$$

with  $c(\varphi) = \iota_c(f)$ .

Now, let  $d : P' \times_{\pi', W, \pi'} P' \rightarrow G$  be the division map for  $P'$ : the map defining  $d(p, p')$  as the unique element of  $G$  such that  $p \cdot d(p, p') = p'$  for any  $p, p' \in P'$  with  $\pi'(p) = \pi'(p')$ . This is a smooth map. For each element  $t \in \mathbb{R}$ , define

$$\tilde{\varphi}_t : P \rightarrow G \quad \tilde{\varphi}_t(p) := d(\varphi(t \cdot p), t \cdot \varphi(p)).$$

By design, this map measures the failure of  $\varphi$  to be equivariant with respect to the action of  $t$ .

Again, recall the interior  $\mathring{W} \subset W$  is an open dense subset of  $W$  and that  $c|_{\mathring{W}}$  functions as the identity (see Remark 2.10). So,  $c(\varphi|_{\mathring{W}}) = \iota_c(f|_{\mathring{W}})$  is  $\mathbb{R}$ -equivariant and we therefore have that, for every  $t \in \mathbb{R}$  and for every  $p \in P|_{\mathring{W}}$ ,  $\tilde{\varphi}_t(p) = e$ , for  $e$  the identity element of  $G$ . As  $\tilde{\varphi}_t$  is continuous and constant on the open dense subset  $P|_{\mathring{W}}$  of  $P$ , it follows that  $\tilde{\varphi}_t$  must be the constant  $e$  on all  $P$  for every  $t \in \mathbb{R}$ . Thus,  $\varphi : \iota_h(P, \omega) \rightarrow \iota_h(P', \omega')$  must be  $\mathbb{R}$ -equivariant.



It follows that  $\varphi$  is actually a map of homogeneous symplectic toric bundles  $\varphi : (P, \omega) \rightarrow (P', \omega')$ . Using the commutativity of the above diagram once more, we have that

$$\iota_c(\mathrm{hc}(\varphi)) = c(\iota_h(\varphi)) = \iota_c(f)$$

Since  $\iota_c$  is faithful, this implies  $\mathrm{hc}(\varphi) = f$ . Hence  $\mathrm{hc}$  is full.  $\square$

We require two more lemmas before we can use Lemma B.11 to prove that  $\mathrm{hc} : \mathrm{HSTB}_\psi \rightarrow \mathrm{STC}_\psi$  is an isomorphism of presheaves.

**Lemma 5.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then any two symplectic toric cones over  $\psi$   $(M, \omega, \pi : M \rightarrow W)$  and  $(M', \omega', \pi' : M' \rightarrow W)$  are locally isomorphic; explicitly, there is an open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $W$  by  $\mathbb{R}$ -invariant open subsets and a collection of isomorphisms

$$\{\varphi_\alpha : (M, \omega, \pi : M \rightarrow W)|_{U_\alpha} \rightarrow (M', \omega', \pi' : M' \rightarrow W)|_{U_\alpha} \in \mathrm{STC}_\psi(U_\alpha) \mid \alpha \in A\}.$$

As the proof involves a number of already known results about the relationship between symplectic toric cones and contact toric manifolds, we relegate the proof of Lemma 5.5 to Appendix A.

**Lemma 5.6.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then the presheaf  $\mathrm{STC}_\psi : \mathrm{Open}_\mathbb{R}(W)^{op} \rightarrow \mathbf{Groupoids}$  is a prestack (see Definition B.4).

*Proof.* To show  $\mathrm{STC}_\psi$  is a prestack, we must show that, for every  $\mathbb{R}$ -invariant open subset  $U$  of  $W$  and for any two symplectic toric cones  $(M, \omega, \pi : M \rightarrow U)$  and  $(M', \omega', \pi' : M' \rightarrow U)$  in  $\mathrm{STC}_\psi(U)$ , the presheaf

$$\underline{\mathrm{Hom}}((M, \omega, \pi), (M', \omega', \pi')) : \mathrm{Open}_\mathbb{R}(U)^{op} \rightarrow \mathbf{Sets} \quad V \mapsto \mathrm{Hom}_{\mathrm{STC}_\psi}((M, \omega, \pi)|_V, (M', \omega', \pi')|_V)$$

is a sheaf of sets. Clearly every morphism  $f : M \rightarrow M'$  is uniquely determined by its restrictions to any open cover, so it remains to show that coherent families of local isomorphisms glue to global maps.

So fix an open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $U$  by  $\mathbb{R}$ -invariant open subsets. Suppose we have a family of isomorphisms of symplectic toric cones

$$\{f_\alpha : (M, \omega, \pi : M \rightarrow U)|_{U_\alpha} \rightarrow (M', \omega', \pi' : M' \rightarrow U)|_{U_\alpha}\}_{\alpha \in A}$$

that are locally coherent; that is, for  $\alpha$  and  $\beta$  with  $U_{\alpha\beta} := U_\alpha \cap U_\beta$  non-empty,  $f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$ . Then clearly there is a unique smooth map  $f : M \rightarrow M'$  such that  $f|_{\pi^{-1}(U_\alpha)} = f_\alpha$  for every  $\alpha$ .

As checking a map is symplectic may be done locally and each  $f_\alpha$  is a symplectomorphism, it follows that  $f$  must be a symplectic map. By design,  $\pi^{-1}(U_\alpha) \subset M$  and  $\pi'^{-1}(U_\alpha) \subset M'$  are both  $(G \times \mathbb{R})$ -invariant and  $f_\alpha$  is equivariant for every  $\alpha$ . Therefore,  $f$  must be  $(G \times \mathbb{R})$ -equivariant. It is clear that, applying the same logic as above, the collection of maps

$$\{f_\alpha^{-1} : (M', \omega', \pi' : M' \rightarrow U)|_{U_\alpha} \rightarrow (M, \omega, \pi : M \rightarrow U)|_{U_\alpha}\}_{\alpha \in A}$$

glue together to a  $(G \times \mathbb{R})$ -equivariant symplectic map

$$f^{-1} : (M', \omega', \pi' : M' \rightarrow U) \rightarrow (M, \omega, \pi : M \rightarrow U)$$

and that this is indeed the inverse to  $f$ . Thus, the  $\{f_\alpha\}_{\alpha \in A}$  glue to a unique isomorphism.

It follows that  $\underline{\mathrm{Hom}}((M, \omega, \pi), (M', \omega', \pi')) : \mathrm{Open}^{op}(U) \rightarrow \mathbf{Sets}$  is a sheaf of sets and therefore  $\mathrm{STC}_\psi$  is a prestack.  $\square$

We may now put together the proof of the following theorem.

**Theorem 5.7.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then  $\mathrm{hc} : \mathrm{HSTB}_\psi \rightarrow \mathrm{STC}_\psi$  is an isomorphism of presheaves. In particular, this means the groupoids  $\mathrm{HSTB}_\psi(W)$  and  $\mathrm{STC}_\psi(W)$  are equivalent.

*Proof.* We have from Lemma 5.6 that  $\mathrm{STC}_\psi$  is a prestack and from Proposition B.8 that  $\mathrm{HSTB}_\psi$  is a stack. To see that  $\mathrm{HSTB}_\psi(U)$  is non-empty for every open  $\mathbb{R}$ -invariant subset  $U$  of  $W$ , note that the trivial principal  $G$ -bundle  $U \times G \rightarrow U$  has a connection 1-form  $A$  with respect to which  $(U \times G, d\langle A, \psi \rangle)$  is a homogeneous symplectic toric bundle. From Lemma 5.5, we have that, for any open  $\mathbb{R}$ -invariant subset  $U$  of  $W$ , any two elements in the groupoid  $\mathrm{STC}_\psi(U)$  are locally isomorphic; in other words,  $\mathrm{STC}_\psi$  is transitive. Finally, from Lemma 5.4, we have that  $\mathrm{hc}_U : \mathrm{HSTB}_\psi(U) \rightarrow \mathrm{STC}_\psi(U)$  is fully faithful for each  $U$ .

Thus,  $\mathrm{HSTB}_\psi$ ,  $\mathrm{STC}_\psi$ , and  $\mathrm{hc}$  satisfy all the hypotheses of Lemma B.11 (also, see Remark B.12) and so we may conclude that  $\mathrm{hc}$  is an isomorphism of presheaves.  $\square$

## 6. CHARACTERISTIC CLASSES FOR SYMPLECTIC TORIC CONES

In this section, we first give characteristic classes for homogeneous symplectic toric bundles over any homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ . Via the isomorphism of presheaves  $\mathrm{hc}$ , these classes then yield characteristic classes for symplectic toric cones taking values in the cohomology group  $H^2(W, \mathbb{Z}_G)$ .

First, we set some notation.

**Notation 6.1.** Given two categories  $\mathcal{C}$  and  $\mathcal{D}$  and a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , denote by the symbols  $\pi_0\mathcal{C}$  and  $\pi_0\mathcal{D}$  the collections of isomorphism classes of  $\mathcal{C}$  and  $\mathcal{D}$  respectively and denote by  $\pi_0F : \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$  the function  $\pi_0F([c]) := [F(c)]$  for each class  $[c] \in \pi_0\mathcal{C}$ . Note  $\pi_0F$  is well-defined as  $F$  is a functor.

**Remark 6.2.** For  $X$  a topological space, suppose  $\mathcal{F} : \mathrm{Open}(X)^{op} \rightarrow \mathrm{Groupoids}$  is a presheaf of groupoids. Then there is a sheaf of sets  $\pi_0\mathcal{F} : \mathrm{Open}(X)^{op} \rightarrow \mathrm{Sets}$  with  $(\pi_0\mathcal{F})(U) := \pi_0(\mathcal{F}(U))$  for every open subset  $U$  of  $X$ . For  $U \subset V$  nested open subsets of  $X$ , the restriction functor  $\rho_{VU} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$  for  $\mathcal{F}$  descends to the function  $\pi_0\rho_{VU} : \pi_0\mathcal{F}(V) \rightarrow \pi_0\mathcal{F}(U)$ . It is easy to check these functions satisfy the necessary requirements of restriction functions for  $\pi_0\mathcal{F} : \mathrm{Open}(X)^{op} \rightarrow \mathrm{Sets}$ .

Now we relate  $\mathrm{HSTB}_\psi$  to an easier presheaf of groupoids to classify.

**Definition 6.3.** For  $W$  a manifold with corners with a free  $\mathbb{R}$ -action, let  $\mathrm{BG}_{\mathbb{R}} : \mathrm{Open}_{\mathbb{R}}(W)^{op} \rightarrow \mathrm{Groupoids}$  be the presheaf of groupoids so that, for every  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ ,  $\mathrm{BG}_{\mathbb{R}}(U)$  is the groupoid of principal  $G$ -bundles over  $U$  with morphisms isomorphisms of principal  $G$ -bundles.

We need the following theorem, well-known in the case of topological spaces.

**Theorem 6.4.** Let  $\pi : E \rightarrow B$  be a principal  $G$ -bundle of manifolds with corners and, for a manifold with corners  $X$ , let  $f_0 : X \rightarrow B$  and  $f_1 : X \rightarrow B$  be two smoothly homotopic maps. Then the pullbacks  $f_0^*(E)$  and  $f_1^*(E)$  are isomorphic as principal  $G$ -bundles.

*Proof.* First, assume  $\pi$  is a bundle of manifolds and  $X$  is also a manifold. Let  $H : X \times [0, 1] \rightarrow B$  be the hypothesized homotopy with  $H(x, 0) = f_0(x)$  and  $H(x, 1) = f_1(x)$ . Then the flow

of the lift horizontal lift of the vector field  $\frac{d}{dt}$  on  $X \times [0, 1]$  with respect to any connection 1-form on  $E$  induces an isomorphism of principal  $G$ -bundles  $f_0^*E \cong f_1^*E$ . Note this flow exists and is equivariant as it matches the parallel transport of the curves  $t \mapsto H(x, t)$  for each  $x \in X$ .

To conclude the same result for manifolds with corners, it is enough to show parallel transport is well-defined in this case. The horizontal lift of  $\frac{d}{dt}$  still makes sense, but we must be able to show that this lift has a flow. The standard existence argument for parallel transport over a manifold with corners  $M$  applies to this case with the possible exception of a path that intersects the boundary of  $M$ . But, for any curve  $\gamma : [a, b] \rightarrow M$ , and any  $s$  with  $a < s < b$  and  $\gamma(s) \in \delta M$  (i.e., the boundary of  $M$ ),  $\gamma(s)$  must be tangent to  $\delta M$ . It follows that there is no obstruction to the existence of the required flow.  $\square$

**Proposition 6.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then there is a map of presheaves  $R : \text{HSTB}_\psi \rightarrow \text{BG}_{\mathbb{R}}$  so that  $\pi_0 R : \pi_0 \text{HSTB}_\psi \rightarrow \pi_0 \text{BG}_{\mathbb{R}}$  is an isomorphism.

*Proof.* For every  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ , let  $R_U : \text{HSTB}_\psi(U) \rightarrow \text{BG}_{\mathbb{R}}(U)$  be the forgetful functor: for every homogeneous symplectic toric bundle  $(\pi : P \rightarrow U, \omega)$ , let  $R(\pi : P \rightarrow U, \omega) := \pi : P \rightarrow U$ . Since any map of homogeneous symplectic toric bundles  $\varphi$  is, in particular, an isomorphism of principal  $G$ -bundles, it makes sense to define  $R(\varphi) := \varphi$ . It is clear that  $R$  commutes with restrictions and therefore  $R : \text{HSTB}_\psi \rightarrow \text{BG}_{\mathbb{R}}$  is a map of presheaves over  $\text{Open}_{\mathbb{R}}(W)$ .

It remains to be shown that, for every  $\mathbb{R}$ -invariant open subset  $U$  of  $W$ ,

$$(\pi_0 R)_U : \pi_0 \text{HSTB}_\psi(U) \rightarrow \pi_0 \text{BG}_{\mathbb{R}}(U)$$

is a bijection. Since  $\text{BG}_{\mathbb{R}}(U)$  is a groupoid, it is enough to show that  $R_U$  is essentially surjective and, for any two homogeneous symplectic toric bundles  $(\pi : P \rightarrow U, \omega)$  and  $(\pi' : P' \rightarrow U, \omega')$ ,  $R(P, \omega)$  and  $R(P', \omega')$  are isomorphic only if  $(P, \omega)$  and  $(P', \omega')$  are isomorphic.

To begin, fix a principal  $G$ -bundle  $\pi : P \rightarrow U$ . By Proposition 3.9, the  $\mathbb{R}$ -quotient  $q : U \rightarrow U/\mathbb{R}$  is a principal  $\mathbb{R}$ -bundle. Then there exists a slice  $\Sigma$  for the  $\mathbb{R}$  action on  $U$  (the image of a global section of  $q : U \rightarrow U/\mathbb{R}$ ). With respect to this slice,  $U$  is equivariantly isomorphic to  $U/\mathbb{R} \times \mathbb{R}$ , so there is a homotopy  $H : [0, 1] \times U \rightarrow U$  between the identity map on  $U$  and the contraction of  $U$  onto a slice of the  $\mathbb{R}$  action. By Theorem 6.4, this induces an isomorphism of principal  $G$ -bundles between  $P|_\Sigma \times \mathbb{R} \rightarrow U$  and  $P$ . It follows that  $P$  inherits a free  $\mathbb{R}$  action with respect to which  $\pi$  is equivariant. By Proposition 4.6, there is a connection 1-form  $A$  on  $P$  with respect to which  $(\pi : P \rightarrow U, d\langle \psi \circ \pi, A \rangle)$  is a homogeneous unimodular local embedding with  $R_U(\pi : P \rightarrow U, d\langle \psi \circ \pi, A \rangle) = \pi : P \rightarrow U$ .

Now, suppose  $(\pi : P \rightarrow U, \omega)$  and  $(\pi' : P' \rightarrow U, \omega')$  are two homogeneous symplectic toric bundles and that  $\varphi : P \rightarrow P'$  is an isomorphism. By Lemma 4.2,  $\pi$  and  $\pi'$  are  $\mathbb{R}$ -equivariant. Then for any  $p$  in  $P$ ,

$$\pi'(\varphi(t \cdot p)) = \pi(t \cdot p) = t \cdot \pi(p) = t \cdot \pi'(\varphi(p)) = \pi'(t \cdot \varphi(p))$$

So, while  $\varphi$  needn't be  $\mathbb{R}$ -equivariant,  $\varphi(t \cdot p)$  and  $t \cdot \varphi(p)$  must lie in the same fiber of  $\pi'$ .

As before, let  $d : P' \times_U P' \rightarrow G$  be the division map: the smooth map taking each pair  $(p, p')$  with  $\pi'(p) = \pi'(p')$  to the unique element of  $G$  satisfying  $p \cdot d(p, p') = p'$ . As above, let  $\Sigma$  be a slice for the action of  $\mathbb{R}$  on  $U$ . Then there is an  $\mathbb{R}$ -equivariant diffeomorphism

$\phi : P \rightarrow P|_{\Sigma} \times \mathbb{R}$  and for  $\phi^1 : P \rightarrow P|_{\Sigma}$  and  $\phi^2 : P \rightarrow \mathbb{R}$  the corresponding projections, we may define the isomorphism

$$\tilde{\varphi} : P \rightarrow P' \quad p \mapsto \varphi(p) \cdot d(\varphi(p), \phi^2(p) \cdot \varphi(\phi^1(p)))$$

Then, since  $\phi^1(t \cdot p) = \phi^1(p)$  for any  $p$  in  $P$  and  $t$  in  $\mathbb{R}$ , we have

$$\tilde{\varphi}(t \cdot p) = \varphi(t \cdot p) \cdot d(\varphi(t \cdot p), \phi^2(t \cdot p) \cdot \varphi(\phi^1(t \cdot p))) = \phi^2(t \cdot p) \cdot \varphi(\phi^1(p)) = t \cdot \tilde{\varphi}(p)$$

Thus,  $\tilde{\varphi}$  is  $\mathbb{R}$ -equivariant and therefore a  $(G \times \mathbb{R})$ -equivariant isomorphism. So, by Proposition 4.10, the two homogeneous symplectic toric bundles  $(\pi : P \rightarrow U, \omega)$  and  $(\pi' : P' \rightarrow U, \omega')$  are isomorphic.  $\square$

Before we can finish, we need the following well-known theorem.

**Theorem 6.6.** Let  $M$  be a manifold with corners and let  $\mathbf{BG}(M)$  be the category of principal  $G$ -bundles over  $M$  with morphisms isomorphisms of principal  $G$ -bundles. For  $G$  our torus and  $\mathbb{Z}_G$  the integral lattice of  $\mathfrak{g}$  (that is, the kernel of  $\exp : \mathfrak{g} \rightarrow G$ ), the function:

$$c_1 : \pi_0 \mathbf{BG}(M) \rightarrow H^2(M; \mathbb{Z}_G)$$

with  $c_1([P]) := c_1(P)$  the first Chern class of  $P$  is a bijection.

**Remark 6.7.** We may extend the bijection in Theorem 6.6 to an isomorphism of presheaves of sets. Since  $\mathbf{BG} : \mathbf{Open}(M)^{\text{op}} \rightarrow \mathbf{Groupoids}$  is a presheaf of groupoids (in fact, a stack; see Example B.7), as explained in Remark 6.2,  $\pi_0 \mathbf{BG} : \mathbf{Open}(M)^{\text{op}} \rightarrow \mathbf{Sets}$  is a presheaf of sets. Since the first Chern class  $c_1$  is a characteristic class, it commutes with restrictions, and so we may think of the collection of bijections

$$c_1 : \pi_0 \mathbf{BG}(U) \rightarrow H^2(U, \mathbb{Z}_G)$$

as an isomorphism of presheaves of sets.

Now we may classify homogeneous symplectic toric bundles over a homogeneous unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ .

**Proposition 6.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then, for  $H^2(\cdot, \mathbb{Z}_G) : \mathbf{Open}_{\mathbb{R}}(W)^{\text{op}} \rightarrow \mathbf{Sets}$  the presheaf of sets  $U \mapsto H^2(U, \mathbb{Z}_G)$ , there is an isomorphism of presheaves:

$$\text{ch} : \pi_0 \mathbf{HSTB}_{\psi} \rightarrow H^2(\cdot, \mathbb{R}).$$

*Proof.* Recall we have isomorphisms of presheaves  $R : \pi_0 \mathbf{HSTB}_{\psi} \rightarrow \pi_0 \mathbf{BG}_{\mathbb{R}}$  of Proposition 6.5 and  $c_1 : \pi_0 \mathbf{BG} \rightarrow H^2(\cdot, \mathbb{Z}_G)$  from Remark 6.7. Therefore, the composition  $\text{ch} := c_1 \circ R$  is an isomorphism of presheaves.  $\square$

We may now prove our first main classification which we restate for the convenience of the reader.

**Theorem 1.2.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then the set of isomorphism classes of symplectic toric cones over  $\psi$  is in natural bijective correspondence with the cohomology classes  $H^2(W, \mathbb{Z}_G)$ .

*Proof.* This bijective correspondence arises from the composition of isomorphisms of presheaves:  $(\pi_0 \text{hc})^{-1} : \mathbf{STC}_{\psi} \rightarrow \mathbf{HSTB}_{\psi}$  and  $\text{ch} : \mathbf{HSTB}_{\psi} \rightarrow H^2(\cdot, \mathbb{Z}_G)$ .  $\square$

We have an easy corollary.

**Corollary 6.9.** Suppose a symplectic toric cone  $(M, \omega)$  with orbital moment map  $\bar{\mu} : M/G \rightarrow \mathfrak{g}^*$  satisfies  $H^2(M/G, \mathbb{Z}_G) = 0$ . Then  $(M, \omega)$  is  $(G \times \mathbb{R})$ -equivariantly symplectomorphic to every other symplectic toric cone admitting quotient space  $M/G$  and orbital moment map  $\bar{\mu}$ .

## 7. RELATION TO CONTACT TORIC MANIFOLDS

Now, we'll discuss how the classification of symplectic toric cones over a chosen homogeneous unimodular local embedding descends to a classification of contact toric manifolds of a specific type. Strictly speaking, this classification is just an exploitation of the well-known relationship between symplectic cones and co-oriented contact manifolds, described in Appendix A. We follow Lerman [15] in defining contact toric manifolds as co-oriented contact manifolds  $(B, \xi)$  with an effective contact action by a torus  $G$  so that  $2 \dim(G) = \dim(B) + 1$ .

To begin, we define two groupoids of interest.

**Definition 7.1.** Let **STC** be the category of symplectic toric cones: the groupoid with objects symplectic toric cones (see Definition 3.1) and isomorphisms  $(G \times \mathbb{R})$ -equivariant symplectomorphisms. Let **CTM** be the category of contact toric manifolds: the groupoid with objects contact toric manifolds and morphisms co-orientation preserving  $G$ -equivariant contactomorphisms.

A particular example of a symplectic toric cone may be associated to any contact toric manifold.

**Definition 7.2.** Given a contact toric manifold  $(B, \xi)$ , the symplectization  $\xi_+^o$  of  $(B, \xi)$  is the line bundle  $\pi : \xi_+^o \rightarrow B$ . Here,  $\pi$  denotes the restriction of the natural projection  $T^*B \rightarrow B$  to the component of  $\xi^o \setminus 0$  (i.e., the annihilator of  $\xi$  minus its zero section) chosen by the co-orientation of  $(B, \xi)$ . This line bundle inherits the structure of a symplectic toric cone from  $T^*B$ . See Remark A.3 for a more complete explanation of this space.

We will now explicitly describe the aforementioned relationship between symplectic toric cones and contact toric manifolds.

**Theorem 7.3.** There is an equivalence of categories  $\Phi : \text{STC} \rightarrow \text{CTM}$ .

*Proof.* As explained in Appendix A, the  $\mathbb{R}$ -quotient  $B := M/\mathbb{R}$  of a symplectic toric cone  $(M, \omega, \mu)$  inherits a co-oriented contact structure  $\xi$  as well as a contact  $G$ -action. Additionally, any map of symplectic toric cones  $\varphi : (M, \omega, \mu) \rightarrow (M', \omega', \mu')$  descends to a  $G$ -equivariant contactomorphism  $\phi : (B, \xi) \rightarrow (B' := M'/\mathbb{R}, \xi')$ . Then define  $\Phi(M, \omega, \mu) := (B, \xi)$  and  $\Phi(\varphi) := \phi$ .

To see  $\Phi$  is essentially surjective, note that, for any contact toric manifold  $(B, \xi)$ , the symplectization  $\xi_+^o$  is a symplectic toric cone.

To see  $\Phi$  is full, let  $(M, \omega, \mu)$  and  $(M', \omega', \mu')$  be two symplectic cones with  $\Phi(M, \omega, \mu) = (B, \xi)$  and  $\Phi(M', \omega', \mu') = (B', \xi')$ . Suppose then that  $f : (B, \xi) \rightarrow (B', \xi')$  is a co-orientation preserving  $G$ -equivariant contactomorphism. Then, as symplectic cones, a choice of  $G$ -equivariant trivializations  $\phi : M \rightarrow B \times \mathbb{R}$  and  $\phi' : M' \rightarrow B' \times \mathbb{R}$  of  $M$  and  $M'$  as principal  $\mathbb{R}$ -bundles over  $B$  and  $B'$  induces a choice of  $G$ -invariant contact forms  $\alpha$  and  $\alpha'$  such that  $\phi^*(d(e^t \alpha)) = \omega$  and  $\phi'^*(d(e^t \alpha')) = \omega'$ .

Since  $f$  is a co-orientation preserving contactomorphism and  $\alpha$  and  $\alpha'$  lie in the respective conformal classes determined by the co-orientations of  $(B, \xi)$  and  $(B', \xi')$ , we must have that  $f^*\alpha' = e^g\alpha$  for some  $G$ -invariant function  $g$  on  $B$ . It follows that the map

$$\varphi : B \times \mathbb{R} \rightarrow B' \times \mathbb{R} \quad \varphi(b, t) := (f(b), t - g(b)) \quad (7.1)$$

is a  $G$ -equivariant map of symplectic cones between  $(B \times \mathbb{R}, d(e^t\alpha))$  and  $(B' \times \mathbb{R}, d(e^t\alpha'))$ . Therefore,  $\phi'^{-1} \circ \varphi \circ \phi$  is a map of symplectic toric cones from  $(M, \omega, \mu)$  to  $(M', \omega', \mu')$  with  $\Phi(\phi'^{-1} \circ \varphi \circ \phi) = f$ .

Finally, to show  $\Phi$  is faithful, we will show that  $\varphi$  above is *the only* morphism satisfying  $\Phi(\varphi) = f$ . Suppose a map of symplectic toric cones  $h : (M, \omega, \mu) \rightarrow (M', \omega', \mu')$  satisfies  $\Phi(h) = f$ . Then

$$\varphi' := \phi' \circ h \circ \phi^{-1} : (B \times \mathbb{R}, d(e^t\alpha)) \rightarrow (B' \times \mathbb{R}, d(e^t\alpha'))$$

is a symplectomorphism. As it is also a map of principal  $\mathbb{R}$ -bundles covering  $f : B \rightarrow B'$ , there is a smooth map  $\tau : B \rightarrow \mathbb{R}$  with  $\varphi'(b, t) = (f(b), t + \tau(b))$ . We calculate:

$$\varphi'^*(d(e^t\alpha')) = d(e^{t+\tau}f^*\alpha') = d(e^{\tau+g}e^t\alpha)$$

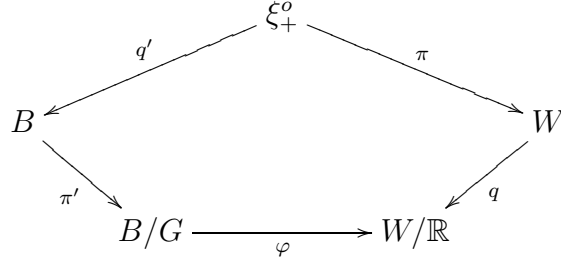
To conclude that  $\varphi'^*(d(e^t\alpha')) = d(e^t\alpha)$ , we must conclude that  $\tau + g = 0$ . Thus, the map  $\varphi$  described in equation (7.1) is the unique map of symplectic toric cones with  $\Phi(\varphi) = f$ .  $\square$

Now, we give a lemma necessary for defining contact toric manifolds over  $\psi$ .

**Lemma 7.4.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $q : W \rightarrow W/\mathbb{R}$  be the  $\mathbb{R}$ -quotient of  $W$ . Suppose  $(B, \xi)$  is a co-oriented contact toric manifold with symplectization  $q' : \xi_+^o \rightarrow B$  for which there is a  $G$ -quotient map  $\pi : \xi_+^o \rightarrow W$  with respect to which  $(\xi_+^o, \pi : \xi_+^o \rightarrow W)$  is a symplectic toric cone over  $\psi$ . Then there is a unique  $G$ -quotient  $\varpi : B \rightarrow W/\mathbb{R}$  for which  $\varpi \circ q' = q \circ \pi$ .

*Proof.* Since the actions of  $G$  and  $\mathbb{R}$  on  $\xi_+^o$  commute, we have a combined action of  $G \times \mathbb{R}$  on  $\xi_+^o$ . Suppose  $N$  is any manifold with corners and  $f : \xi_+^o \rightarrow N$  is a smooth  $(G \times \mathbb{R})$ -invariant map. Then, in particular,  $f$  is  $G$ -invariant, so there is a unique smooth map  $f' : W \rightarrow N$  with  $f' \circ \pi = f$ . Now, since  $\pi$  is  $\mathbb{R}$ -equivariant and  $f$  is  $\mathbb{R}$ -invariant, we may conclude that  $f'$  is  $\mathbb{R}$ -invariant and therefore there exists a unique smooth map  $f'' : W/\mathbb{R} \rightarrow N$  so that  $f' = f'' \circ q$ . We may therefore conclude the composite  $q \circ \pi : \xi_+^o \rightarrow W/\mathbb{R}$  is the  $(G \times \mathbb{R})$ -quotient for  $M$  (as a manifold with corners).

Now, recall that the quotient  $B$  inherits a contact structure  $\xi$  with respect to which  $(B, \xi)$  is contact toric and the quotient  $q'$  is  $G$ -equivariant. Using a symmetric argument as in the previous paragraph, we may conclude that, for  $\pi' : B \rightarrow B/G$  a  $G$ -quotient of  $B$ , the composite  $\pi' \circ q' : M \rightarrow B/G$  is also a  $(G \times \mathbb{R})$ -quotient map for  $M$  (here, we use the fact that  $B/G$  is a manifold with corners as well; see Lemma A.18). It follows that there exists a unique diffeomorphism  $\varphi : B/G \rightarrow W/\mathbb{R}$  so that  $\varphi \circ \pi' \circ q' = q \circ \pi$ .



Then let  $\varpi := \varphi \circ \pi'$ . It follows from above that  $\varpi$  is a  $G$ -quotient for  $W/\mathbb{R}$ . Additionally, since  $q \circ \pi$  is  $\mathbb{R}$ -invariant, it follows from the universal property of the  $\mathbb{R}$ -quotient  $q' : \xi_+^o \rightarrow B$  that  $\varpi : B \rightarrow W/\mathbb{R}$  is the unique map satisfying  $\varpi \circ q' = q \circ \pi$ .  $\square$

Now, we make the following definition of contact toric manifolds over  $\psi$ .

**Definition 7.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and fix an  $\mathbb{R}$ -quotient  $q : W \rightarrow W/\mathbb{R}$ . Then for  $(B, \xi)$  a contact toric manifold with for symplectization  $q' : \xi_+^o \rightarrow B$ , a **contact toric manifold over  $\psi$**  is a tuple  $(B, \xi, \pi : \xi_+^o \rightarrow W, \varpi : B \rightarrow W/\mathbb{R})$ , where  $\pi$  is a  $G$ -quotient with respect to which  $(\xi_+^o, \pi)$  is a symplectic toric cone over  $\psi$  and  $\varpi : B \rightarrow W/\mathbb{R}$  is the unique  $G$ -quotient satisfying  $\varpi \circ q' = q \circ \pi$  of Lemma 7.4.

The **groupoid of contact toric manifolds over  $\psi$**  is the groupoid with objects contact toric manifolds over  $\psi$  and morphisms

$$f : (B, \xi, \pi : \xi_+^o \rightarrow W, \varpi : B \rightarrow W/\mathbb{R}) \rightarrow (B', \xi', \pi' : \xi'^o_+ \rightarrow W, \varpi' : B' \rightarrow W/\mathbb{R})$$

$G$ -equivariant co-orientation preserving contactomorphisms satisfying  $\varpi' \circ f = \varpi$ . This is denoted by  $\mathbf{CTM}_\psi(W/\mathbb{R})$ .

**Remark 7.6.** As in the case of symplectic toric cones, contact toric manifolds over  $\psi$  form a presheaf of groupoids. Given an open subset  $U$  of  $W/\mathbb{R}$  and a contact toric manifold over  $\psi$   $(B, \xi, \pi : \xi_+^o \rightarrow W, \varpi : B \rightarrow W/\mathbb{R})$ , one may check that

$$(B, \xi, \pi : \xi_+^o \rightarrow W, \varpi : B \rightarrow W/\mathbb{R})|_U := (\varpi^{-1}(U), \xi|_{\varpi^{-1}(U)}, \pi|_{\pi^{-1}(q^{-1}(U))}, \varpi|_{\varpi^{-1}(U)})$$

gives a well-defined element of  $\mathbf{CTM}_\psi(U) := \mathbf{CTM}_{\psi|_{q^{-1}(U)}}(U)$  and, since morphisms of  $\mathbf{CTM}_\psi(W/\mathbb{R})$  must preserve the quotients of contact toric manifolds to  $W/\mathbb{R}$ , restrictions of maps of  $\mathbf{CTM}_\psi(W)$  are well-defined as well. Therefore, we have a presheaf of groupoids:

$$\mathbf{CTM}_\psi : \mathbf{Open}(W/\mathbb{R})^{op} \rightarrow \mathbf{Groupoids}$$

We now define two important functors that will motivate the above definition.

**Definition 7.7.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and fix an  $\mathbb{R}$ -quotient  $q : W \rightarrow W/\mathbb{R}$ . Then let  $\iota_s : \mathbf{STC}_\psi(W) \rightarrow \mathbf{STC}$  and  $\iota_c : \mathbf{CTM}_\psi(W) \rightarrow \mathbf{CTM}$  be the forgetful functors “forgetting” the quotient structures of each source category.

By design,  $\mathbf{CTM}_\psi(W)$  is related to  $\mathbf{STC}_\psi(W)$  via  $\Phi$  and the functors  $\iota_s$  and  $\iota_c$ .

**Proposition 7.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and fix an  $\mathbb{R}$ -quotient  $q : W \rightarrow W/\mathbb{R}$ . Then  $\iota_c(\mathbf{CTM}_\psi(W/\mathbb{R})) = \iota_s(\mathbf{STM}_\psi(W))$ . Thus, there is a natural bijection between the isomorphism classes of  $\mathbf{STC}_\psi(q^{-1}(U))$  and  $\mathbf{CTM}_\psi(U)$  for each open subset  $U$  of  $W/\mathbb{R}$ .

*Proof.* Given a contact toric manifold over  $\psi$   $(B, \xi, \pi : \xi_+^o \rightarrow W, \varpi : B \rightarrow W/\mathbb{R})$ ,  $(\xi_+^o, \pi : \xi_+^o)$  is a symplectic toric cone over  $\psi$ . It is easy to check that, since a map of symplectic toric manifolds  $f$  must preserve the  $G$ -quotient maps to  $W$  of the source and target of  $f$ ,  $\Phi(f)$  must preserve the  $G$ -quotient maps to  $W/\mathbb{R}$  of the source and target of the corresponding underlying contact toric manifolds.

It is also easy to check that, for  $U \subset V$  open subsets of  $W/\mathbb{R}$ , restriction of a contact toric manifold over  $\psi|_{q^{-1}(V)}$  to  $U$  corresponds to restriction of symplectic toric cones over  $\psi|_{q^{-1}(V)}$  to  $q^{-1}(U)$ .  $\square$

We can now prove our second classification result.

**Theorem 1.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then there is a natural bijective correspondence between the set of isomorphism classes contact toric manifolds  $(B, \xi)$  with symplectizations  $\xi_+^o$  admitting a  $G$ -quotient map  $\pi : \xi_+^o \rightarrow W$  with orbital moment map  $\psi$  and the cohomology classes  $H^2(W, \mathbb{Z}_G)$ .

*Proof.* The isomorphism classes of contact toric manifolds with symplectizations admitting the structure of a symplectic toric cone over  $\psi$  are exactly the image under  $\iota_c$  of  $\mathbf{CTM}_\psi(W/\mathbb{R})$  in  $\mathbf{CTM}$ . By Proposition 7.8, this corresponds via  $\Phi$  to the image of  $\mathbf{STC}_\psi(W)$  in  $\mathbf{STC}$  under the functor  $\iota_s$ . Since  $\iota_c$  and  $\iota_s$  are faithful, it follows by Theorem 1.2 that the isomorphism classes of  $\mathbf{CTM}_\psi(W/\mathbb{R})$  are in natural bijective correspondence with  $H^2(W, \mathbb{Z}_G)$ .  $\square$

## Part II: Classifying symplectic toric stratified spaces with isolated singularities

The goal of this section is to describe and classify symplectic toric stratified spaces with isolated singularities. To begin, we describe in Section 8 *singular symplectic toric cones*: these are symplectic toric cones with an added point at infinity. These spaces are important, as they will serve as a model for symplectic toric stratified spaces with isolated singularities.

In Section 9, We define and describe symplectic toric stratified spaces with isolated singularities. These are (roughly) stratified spaces with torus actions locally modelled on singular symplectic toric cones (see Definition 9.1). We will see that, for such a space  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$ , the topological quotient  $X/G$  of any symplectic toric stratified space with isolated singularities  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  inherits the structure of a *cornered stratified space with isolated singularities*; essentially, a stratified space for which the stratum are allowed to be manifolds with corners. Furthermore, the moment map  $\mu : X \rightarrow \mathfrak{g}^*$  descends to a *stratified unimodular local embedding*  $\bar{\mu} : X/G \rightarrow \mathfrak{g}^*$  (see Definition 9.2). This is the continuous extension of a unimodular local embedding to a cornered stratified space with an additional local property near each singularity.

As any isomorphic symplectic toric stratified spaces must have the same orbit space and orbital moment map, we group the spaces with orbital moment map the stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  together into a groupoid  $\mathbf{STSS}_\psi(W)$ , the *groupoid of symplectic toric stratified spaces over  $\psi$* . The morphisms of this groupoid are exactly the  $G$ -equivariant isomorphisms (i.e., strata preserving homeomorphisms descending to symplectomorphisms on the open dense strata) preserving  $\psi$ . As in the case of symplectic toric cones, we may form a presheaf with these groupoids:

$$\mathbf{STSS}_\psi : \mathbf{Open}(W)^{op} \rightarrow \mathbf{Groupoids}$$



In Section 10, we define *conical symplectic toric  $G$ -bundles over  $\psi$* . These are principal  $G$ -bundles  $\pi : P \rightarrow W_{\text{reg}}$  over the open dense stratum  $W_{\text{reg}}$  of  $W$  with a  $G$ -invariant symplectic form for which  $\psi \circ \pi$  is a moment map satisfying a special “conical” condition (see Definition 10.1). Together with  $G$ -equivariant symplectomorphisms, these form a groupoid  $\text{CSTB}_\psi(W)$ , the *groupoid of conical symplectic toric  $G$ -bundles over  $\psi$* . As in the case of homogeneous symplectic toric bundles, the collection of groupoids  $\text{CSTB}_{\psi|_U}(U)$  forms a presheaf of groupoids

$$\text{HSTB}_\psi : \text{Open}(W)^{op} \rightarrow \text{Groupoids}$$

In Section 11, we build a map of presheaves  $\tilde{c} : \text{CSTB}_\psi \rightarrow \text{STSS}_\psi$  adapted from the equivalence of categories  $c$  presented by Karshon and Lerman. As in the case of  $c$  and  $\text{hc}$  of Part I, we are also able to show that  $\tilde{c}$  is an isomorphism of presheaves (Theorem 11.10).

In Section 6, we show in Proposition 12.4 that isomorphism classes of conical symplectic toric bundles are determined both by their structure as principal  $G$ -bundles and by a so-called horizontal class. This mirrors the case of Karshon and Lerman where horizontal classes took the form of cohomology classes on the base manifold with corners  $W$ . In the case of conical symplectic toric bundles, however, there are restrictions on the allowed cohomology classes; namely classes of forms called (for the purposes of this paper) *good forms*: forms on the open dense manifold piece  $W_{\text{reg}}$  of a cornered stratified space  $W$  that are exact in deleted neighborhoods of each singularity of  $W$  (see Definition 12.1). We denote the subspace of all classes of good forms by  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$ . We may then finally use this result and the isomorphism  $\tilde{c}$  to prove our main classification theorem (Theorem 1.1): the bijection between the isomorphism classes of  $\text{STSS}_\psi(W)$  and the cohomology classes  $H^2(W_{\text{reg}}, \mathbb{Z}_G) \times \mathcal{C}$ .

We finish the section by showing that the subspace  $\mathcal{C}$  can be identified as the image of the relative de Rham cohomology group  $H^2(W_{\text{reg}}, \overline{W})$  (as from Bott and Tu, [3]) under a natural inclusion map for an appropriately chosen subset  $\bar{W}$  of  $W_{\text{reg}}$ . This group and image can be computed using the long exact sequence associated to a pair.

In Section 13, we provide some examples of the use of all our main classification theorems. We also provide some explanation of how our classification of contact toric manifolds is an extension of the result of Lerman [15] in the compact connected case as well as an explanation of how our classification of symplectic toric stratified spaces is an extension of the result of Burns, Guillemin, and Lerman [5] again in the compact connected case.

## 8. SINGULAR SYMPLECTIC TORIC CONES

Symplectic toric stratified spaces with isolated singularities are symplectic toric manifolds except on a discrete set of isolated singularities fixed by the torus  $G$ . These singularities have neighborhoods modeled by neighborhoods of  $-\infty$  of singular symplectic toric cones. We will make this precise with a series of definitions.

**Definition 8.1.** Let  $L$  be a manifold (possibly with corners). Then the **open cone on  $L$** , denoted  $c(L)$ , is the topological space  $(L \times [-\infty, \infty))/(L \times \{-\infty\})$ . Here,  $[-\infty, \infty)$  is the topological space given by compactifying  $\mathbb{R}$  at one end and is homeomorphic to  $[0, \infty)$  (or any half closed interval, for that matter). We denote by  $*$  the point of the cone (i.e., the image of  $\{-\infty\} \times L$  in  $c(L)$  under the quotient).

While this convention for a cone is a bit awkward, it fits the convention for symplectic cones nicely and is more convenient in the long run.

What follows is a definition for stratified spaces with isolated singularities. To model both symplectic toric stratified spaces and their quotients, we assume that stratified spaces with isolated singularities may be modeled on cones of either manifolds or manifolds with corners (we distinguish the latter case with the name *cornered stratified spaces*).

**Definition 8.2.** A stratified space with isolated singularities is any stratified space  $X$  such that, with the exception of the open, dense stratum, all other strata are zero dimensional. More concretely, it is a Hausdorff topological space  $X$  with a partition  $X = X_{\text{reg}} \sqcup_{\alpha \in I} \{x_\alpha\}$  such that  $X_{\text{reg}}$  is a manifold and, for each  $x_\alpha$ , there exists a neighborhood  $U_\alpha$  of  $x_\alpha$  in  $X$ , a compact manifold  $L_\alpha$ , and an embedding  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$  such that

- $\varphi_\alpha(x_\alpha) = *$  (i.e.,  $\varphi_\alpha$  maps  $x_\alpha$  to the cone point of  $c(L_\alpha)$ ); and
- $\varphi_\alpha$  restricts to a diffeomorphism between  $X_{\text{reg}} \cap U_\alpha$  and its image in  $c(L_\alpha) \setminus \{*\} \cong L \times \mathbb{R}$

Formally, this data will be represented by the pair  $(X, X_{\text{reg}} \sqcup_{\alpha \in I})$  though informally, the partition may be suppressed.

Call a choice of link  $L_\alpha$ , neighborhood  $U_\alpha$ , and embedding  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$  a **local structure datum** for  $x_\alpha$ .

A **cornered stratified space with isolated singularities** is a stratified space for which the links of the singularities may instead be compact manifolds with corners.

The open dense stratum of a stratified space with isolated singularities  $X$  will always be designated  $X_{\text{reg}}$  and, for the purposes of this paper, will be called the **regular part** of  $X$ .

A map of stratified spaces with isolated singularities is a continuous map

$$f : (X, X_{\text{reg}} \sqcup_{\alpha \in A} \{x_\alpha\}) \rightarrow (X', X'_{\text{reg}} \sqcup_{\beta \in B} \{x'_\beta\})$$

so that  $f(X_{\text{reg}}) \subset X'_{\text{reg}}$  and for every  $\alpha \in A$ ,  $f(x_\alpha) = x'_\beta$  for some  $\beta \in B$ . Such a map is an **isomorphism of stratified spaces with isolated singularities** if it is a homeomorphism (it follows that  $f^{-1}$  is a map of stratified spaces since  $f$  is bijective and a map of stratified spaces).

**Remark 8.3.** Note that, for any (cornered) stratified space  $W$ , an open subset  $U \subset W$  also inherits the structure of a (cornered) stratified space. The open dense part of  $U$  with respect to this structure is exactly the intersection  $U_{\text{reg}} = W_{\text{reg}} \cap U$ .

As a particular important example of a stratified space with isolated singularities note that, for any compact manifold  $L$ , its open cone  $c(L)$  is a stratified space with one isolated singularity. If  $L$  is a compact manifold with corners, then  $c(L)$  is a cornered stratified space with one isolated singularity.

From any symplectic cone, we can build a stratified space by adding a point at  $-\infty$ .

**Definition 8.4.** Given a symplectic cone  $(M, \omega)$ , a **neighborhood of  $-\infty$**  is any open subset  $U$  closed under the action of any negative element of  $\mathbb{R}$ . More precisely, for each real  $\lambda \leq 0$  and for  $\rho_\lambda : M \rightarrow M$  the  $\mathbb{R}$  action diffeomorphism induced by  $\lambda$ ,  $\rho_\lambda(V) \subset V$ .

We now define singular symplectic toric cones.

**Definition 8.5.** A **singular symplectic cone (with corners)** is a (cornered) stratified space with one isolated singularity  $X = X_{\text{reg}} \sqcup \{x_0\}$  together with a symplectic form  $\omega \in \Omega^2(X_{\text{reg}})$  such that

- $X_{\text{reg}}/\mathbb{R}$  is compact;
- $(X_{\text{reg}}, \omega)$  is a symplectic cone; and

- every neighborhood  $U$  of the cone point  $x_0$  of  $X$  contains a neighborhood of  $-\infty$  of  $(X_{\text{reg}}, \omega)$ .

We will soon show that singular symplectic toric cones have a particular important model case, but we first need the following simple structural lemma.

**Lemma 8.6.** Let  $L$  be a compact manifold (with corners). Then any neighborhood  $U$  of the cone point  $*$  of  $c(L)$  contains a neighborhood of the form  $L \times (-\infty, \epsilon) \sqcup \{*\}$ , for some  $\epsilon \in \mathbb{R}$ .

*Proof.* Recall that  $c(L)$  is defined as the quotient  $\pi : L \times [-\infty, \infty) \rightarrow c(L)$ , with  $\pi^{-1}(*) = L \times \{-\infty\}$ . Consequently,  $\pi^{-1}(U)$  is an open subset of  $L \times [-\infty, \infty)$  containing  $L \times \{-\infty\}$ . As  $L$  is compact, it follows there exists  $\epsilon$  such that  $L \times [-\infty, \epsilon) \subset \pi^{-1}(U)$ .  $\square$

Now, we show that every symplectic toric cone has a simple (but important) model as a topological space.

**Proposition 8.7.** Let  $(X = X_{\text{reg}} \sqcup \{x_0\}, X_{\text{reg}})$  be a singular symplectic cone. Then, for  $\mathbb{R}$ -quotient  $L := X_{\text{reg}}/\mathbb{R}$ , every trivialization  $\varphi : X_{\text{reg}} \rightarrow L \times \mathbb{R}$  of  $X_{\text{reg}}$  as a principal  $\mathbb{R}$ -bundle admits an extension to a homeomorphism  $\tilde{\varphi} : X \rightarrow c(L)$ .

Suppose  $(X, \omega)$  is a singular symplectic cone and that  $\varphi : X_{\text{reg}} \rightarrow L \times \mathbb{R}$  is the required trivialization extending to a homeomorphism from  $X$  to  $c(L)$ . Then any other trivialization  $\phi : X_{\text{reg}} \rightarrow L \times \mathbb{R}$  extends to a homeomorphism from  $X$  to  $c(L)$  as well.

*Proof.*  $\varphi$  extends to a bijection  $\tilde{\varphi} : X \rightarrow c(L)$  taking  $x_0$  in  $X$  to  $*$  in  $c(L)$ . It is easy to check using Lemma 8.6 that the neighborhoods of  $-\infty$  of  $X_{\text{reg}}$  map to deleted neighborhoods of  $*$  in  $c(L)$  and visa versa. Thus,  $\tilde{\varphi}$  is continuous, as is the inverse  $\tilde{\varphi}^{-1}$ .  $\square$

From any symplectic cone, we can construct a singular symplectic cone.

**Proposition 8.8.** Any symplectic cone  $(M, \omega)$  over compact base  $L = M/\mathbb{R}$  extends to a singular symplectic cone.

*Proof.* Define the topological space  $\tilde{M}$  as follows: as a set, it is simply the disjoint union  $M \sqcup \{*\}$ , for the point  $*$  representing our (soon to be) cone point.  $\tilde{M}$  is then given the topology generated by sets of the form:

- (1)  $U$ , an open subset of  $M$
- (2)  $V \sqcup \{*\}$ , where  $V \subset M$  is a neighborhood of  $-\infty$

More succinctly, we topologize the set  $\tilde{M}$  by specifying that all open subsets of  $M$  closed under negative translation (the neighborhoods of  $-\infty$ ) are in fact open neighborhoods of the singular point  $*$ . It is clear that, by definition,  $(\tilde{M}, \omega)$  is a singular symplectic toric cone.  $\square$

For now, this proposition will be enough to give us explicit examples of symplectic stratified spaces. It will later be used to construct the functor  $\tilde{\mathfrak{c}}$ , giving a natural way to take a conical principal toric bundle, locally modeled on (non-singular) symplectic cones, to a symplectic stratified space, locally modeled on singular symplectic cones.

From here forward, fix a torus  $G$  with Lie algebra  $\mathfrak{g}$ .

**Definition 8.9.** A singular symplectic toric cone is a singular symplectic cone  $X = X_{\text{reg}} \sqcup \{x_0\}$  with form  $\omega \in \Omega^2(X_{\text{reg}})$  and continuous map  $\mu : X \rightarrow \mathfrak{g}^*$ , admitting an action of torus  $G$  such that

- $G$  fixes the point  $x_0$  and restricts to a smooth action on  $X_{\text{reg}}$ ; and
- The action of  $G$  on  $X_{\text{reg}}$  makes the symplectic cone  $(X_{\text{reg}}, \omega)$  a symplectic toric cone for which  $\mu|_{X_{\text{reg}}}$  is the homogeneous moment map of  $(X_{\text{reg}}, \omega)$  (see Definition 3.1).

We represent this data as the triple  $(X, \omega, \mu)$ .

**Remark 8.10.** For singular symplectic toric cone  $(X = X_{\text{reg}} \sqcup \{x_0\}, \omega, \mu : X \rightarrow \mathfrak{g}^*)$ , as  $\mu|_{X_{\text{reg}}}$  is homogeneous, it follows from the continuity of  $\mu$  that  $\mu(x_0) = 0$ .

It will be important later to understand the structure of quotients of singular symplectic toric cones.

**Lemma 8.11.** Let  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  be a singular symplectic toric cone. Then for  $B = X_{\text{reg}}/\mathbb{R}$ ,  $X/G$  is a cornered stratified space with link  $B/G$ .

*Proof.* It is known that  $B$  has a natural contact structure  $\xi$  and the action of  $G$  on  $X_{\text{reg}}$  descends to a contact toric action on  $(B, \xi)$ . By Proposition 8.7, any trivialization  $\phi : X_{\text{reg}} \rightarrow B \times \mathbb{R}$  extends to a homeomorphism  $\tilde{\varphi} : X \rightarrow c(B)$ . Furthermore, by Proposition A.10, we may choose  $\varphi$  to be a  $G$ -equivariant trivialization.

As the actions of  $G$  and  $\mathbb{R}$  commute and  $\tilde{\varphi}$  is  $G$ -equivariant,  $\tilde{\varphi}$  descends to a homeomorphism  $\bar{\varphi} : X/G \rightarrow c(B/G)$ . By Lemma A.18,  $B/G$  is a manifold with corners and therefore  $\bar{\varphi}$  gives local trivialization data for the singularity  $x_0$  of  $X$  as a cornered stratified space.  $\square$

As in the case of symplectic cones, symplectic toric cones admit trivialization independent extensions to singular symplectic toric cones.

**Proposition 8.12.** Every symplectic toric cone  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  over compact base  $L = M/\mathbb{R}$  extends to a singular symplectic toric cone.

*Proof.* First, Proposition 8.8 tells us how to transform  $(M, \omega)$  into a singular symplectic cone  $(\tilde{M}, \omega)$ . The toric action descends to a contact toric  $G$  action on  $L$  (see Appendix A). With respect to this  $G$  action, we can pick a  $G$ -equivariant trivialization of  $M$  as a principal  $\mathbb{R}$ -bundle  $\varphi : M \rightarrow L \times \mathbb{R}$  (see Proposition A.10) which extends to a homeomorphism  $\tilde{\varphi} : \tilde{M} \rightarrow c(L)$ .

As each set of the form  $L \times (-\infty, \epsilon)$  is  $G$ -invariant, it follows that every neighborhood of  $-\infty$  for  $M$  contains a  $G$ -invariant neighborhood of  $-\infty$ . Thus, the action of  $G$  on  $M$  extends to a continuous action  $\rho : G \times \tilde{M} \rightarrow \tilde{M}$  on  $\tilde{M}$  fixing the singular point. Then, since for any  $G$ -invariant neighborhood of  $-\infty$   $V$  we have that  $\rho^{-1}(V \sqcup \{*\}) = G \times V$ , it follows from the observation above that  $\rho$  is continuous.

Finally, note that, since  $\mu$  is smooth and homogeneous, it follows we can continuously extend  $\mu$  to  $\tilde{\mu} : \tilde{M} \rightarrow \mathfrak{g}^*$  by defining  $\tilde{\mu}(*) := 0$ .  $\square$

To finish this section, we prove that any isomorphism of symplectic toric cones extends to an isomorphism between their extensions as singular symplectic toric cones.

**Lemma 8.13.** Let  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  and  $(X', \omega', \mu' : X' \rightarrow \mathfrak{g}^*)$  be two singular symplectic toric cones for which there is an isomorphism of symplectic toric cones

$$f : (X_{\text{reg}}, \omega, \mu|_{X_{\text{reg}}}) \rightarrow (X'_{\text{reg}}, \omega', \mu'|_{X'_{\text{reg}}}).$$

Then  $f$  extends to an isomorphism of singular symplectic toric cones.

*Proof.* Since  $f$  is  $(G \times \mathbb{R})$ -equivariant,  $f$  must take  $G$ -invariant neighborhoods of  $-\infty$  in  $(X_{\text{reg}}, \omega)$  to  $G$ -invariant neighborhoods of  $-\infty$  in  $(X'_{\text{reg}}, \omega')$ .  $f^{-1}$  satisfies the same property and, as in the proofs above, we may conclude that  $f$  and  $f^{-1}$  extend to maps on the respective symplectic toric cones.  $\square$

## 9. SYMPLECTIC TORIC STRATIFIED SPACES WITH ISOLATED SINGULARITIES

After all the work of the previous section, we are finally ready to give a definition of symplectic toric stratified spaces with isolated singularities. Recall we've fixed a torus  $G$  with lie algebra  $\mathfrak{g}$ .

**Definition 9.1.** A symplectic toric stratified space with isolated singularities is a stratified space with isolated singularities  $(X, X_{\text{reg}} \sqcup_{\alpha \in I} \{x_\alpha\})$  with a symplectic form  $\omega \in \Omega^2(X_{\text{reg}})$ , a continuous map  $\mu : X \rightarrow \mathfrak{g}^*$ , and an action of torus  $G$  such that  $G$  fixes each  $x_\alpha$  and restricts to a smooth, toric action on  $(X_{\text{reg}}, \omega)$  with moment map  $\mu|_{X_{\text{reg}}} : X_{\text{reg}} \rightarrow \mathfrak{g}^*$ . Furthermore, for each  $x_\alpha$ , we require that there exist a  $G$ -invariant neighborhood  $U$  of  $x_\alpha$  in  $X$ , a toric singular symplectic cone  $(C, \omega, \nu : C \rightarrow \mathfrak{g}^*)$ , a  $G$ -invariant neighborhood  $V$  of the cone point of  $C$ , and a  $G$ -equivariant homeomorphism  $\varphi : U \rightarrow V$  such that

- $\varphi(x_\alpha) = *$  (for  $*$  the cone point of  $C$ );
- $\varphi$  restricts to a symplectomorphism between  $U_{\text{reg}}$  and  $V_{\text{reg}}$ ; and
- $\mu|_U = \nu \circ \varphi + \mu(x_\alpha)$ .

These objects are represented as the triple  $(X, \omega, \mu)$  (with the partition of  $X$  left implicit).

We still call  $\mu$  a moment map for the full stratified space. Indeed, one may think of  $\mu$  as a map of stratified spaces, serving as a trivial moment map to each zero dimensional symplectic manifold  $\{x_\alpha\}$ .

As explained in Proposition 8.12, any symplectic toric cone with homogeneous moment map can be extended to a singular symplectic cone; these serve for now as our only example of a symplectic toric stratified space with isolated singularities. More exotic examples are discussed in Section 13.

As in [13], symplectic toric stratified spaces are grouped together together by orbital moment map. To make sense of this, it is important to first understand what form their quotients and orbital moment maps take.

**Definition 9.2.** Let  $(W, W_{\text{reg}} \sqcup_{\alpha \in I} \{w_\alpha\})$  be a cornered stratified space with isolated singularities. Then a continuous map  $\psi : W \rightarrow \mathfrak{g}^*$  is a **stratified unimodular local embedding** if

- $\psi|_{W_{\text{reg}}}$  is a unimodular local embedding; and
- For each  $\alpha$ , there exists a local trivialization datum  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$  for  $w_\alpha$  in  $W$  and a homogeneous unimodular local embedding (see Definition 3.2)  $\phi_\alpha : L_\alpha \times \mathbb{R} \rightarrow \mathfrak{g}^*$  such that  $\psi|_{U_{\alpha \text{reg}}} = \phi_\alpha \circ \varphi_\alpha + \psi(w_\alpha)$ , where  $\phi_\alpha$  is homogeneous with respect to the action by translation on  $L_\alpha \times \mathbb{R}$ .

We will call the piece of local trivialization datum  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$  as above a **homogeneous local trivialization datum**.

**Proposition 9.3.** Suppose  $(X, X_{\text{reg}} \sqcup_{\alpha \in I} \{x_\alpha\})$  is a stratified space with isolated singularities and that  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  is a symplectic toric stratified space with isolated singularities. Then  $X/G$  is a cornered stratified space with isolated singularities and, for quotient map  $\pi : X \rightarrow X/G$ , the unique map  $\bar{\mu} : X/G \rightarrow \mathfrak{g}^*$  satisfying  $\bar{\mu} \circ \pi = \mu$  is a stratified unimodular local embedding.

*Proof.* As  $(X_{\text{reg}}, \omega, \mu|_{X_{\text{reg}}})$  is a symplectic toric manifold, it is already known that  $X_{\text{reg}}/G$  is a manifold with corners and  $\bar{\mu}|_{X_{\text{reg}}/G}$  is a unimodular local embedding (see Proposition 2.3). It remains to show that there exists local structure data for each  $[x_\alpha] \in X/G$  so that  $\psi$  factors through a homogenous unimodular local embedding.

We have already by definition that, for any singular point  $x_\alpha$ , there is a  $G$ -invariant neighborhood  $U_\alpha$  together with a  $G$ -equivariant embedding  $\varphi_\alpha : U_\alpha \rightarrow C_\alpha$  of  $U_\alpha$  onto a neighborhood of the cone point of a singular symplectic toric cone  $(C_\alpha, \omega_\alpha, \nu_\alpha : C_\alpha \rightarrow \mathfrak{g}^*)$  so that  $\mu|_{U_\alpha} = \nu_\alpha \circ \varphi_\alpha + \mu(x_\alpha)$ . Thus,  $\varphi_\alpha$  descends to an embedding of  $\bar{\varphi}_\alpha : U_\alpha/G \rightarrow C_\alpha/G$ . From Lemma 8.11, we have that  $C_\alpha/G$  is isomorphic to  $c(L_\alpha)$  for a compact manifold with corners  $L_\alpha$ . Therefore,  $\bar{\varphi}_\alpha$  is a local structure datum for  $X$ .

It is easy to check via the universal properties of quotients that  $\bar{\mu}|_{U_\alpha/G} = \bar{\nu}_\alpha \circ \bar{\varphi}_\alpha + \mu(x_\alpha)$ , for  $\bar{\nu}_\alpha : C_\alpha/G \rightarrow \mathfrak{g}^*$  defined in the same fashion as  $\bar{\mu}$ . As shown in Proposition 3.4,  $\bar{\nu}_\alpha|_{C_\alpha \setminus \{*\}}$  is a homogeneous unimodular local embedding with respect to the action of  $\mathbb{R}$  on  $C_\alpha/G$  descending from the action on  $C_\alpha$ . It is clear that, with respect to the identification  $C_\alpha/G \cong L_\alpha \times \mathbb{R} \subset c(L_\alpha)$ , this  $\mathbb{R}$  action corresponds to the action by translation on  $L_\alpha \times \mathbb{R}$ . Therefore,  $\bar{\varphi}_\alpha$  is our required local structure datum.  $\square$

We can now define our main category of interest.

**Definition 9.4.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding for  $W$  a cornered stratified space with isolated singularities. A **symplectic toric stratified space over  $\psi$**  is a symplectic toric stratified space  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  together with a  $G$ -quotient map  $\pi : X \rightarrow W$  so that  $\psi \circ \pi = \mu$ . This data will be represented by the triple  $(X, \omega, \pi : X \rightarrow W)$ .

A **map of symplectic toric stratified spaces over  $\psi$**  between  $(X, \omega, \pi : X \rightarrow W)$  and  $(X', \omega', \pi' : X' \rightarrow W)$  is a  $G$ -equivariant isomorphism of stratified spaces  $\varphi : X \rightarrow X'$  that restricts to a symplectomorphism between  $(X_{\text{reg}}, \omega)$  and  $(X'_{\text{reg}}, \omega')$  and satisfies  $\pi' \circ \varphi = \pi$ .

The **groupoid of symplectic toric stratified spaces over  $\psi$**   $\text{STSS}_\psi(W)$  is the groupoid with objects and morphisms as described above.

**Remark 9.5.** Note that for any open subset  $U \subset W$ ,  $\psi|_U$  is also a stratified unimodular local embedding. Therefore, it makes sense to define the presheaf

$$\text{STSS}_\psi : \text{Open}(W)^{\text{op}} \rightarrow \text{Groupoids} \quad U \mapsto \text{STSS}_{\psi|_U}(U).$$

Here, for  $V \subset U$  open subsets, restriction is defined as

$$(X, \omega, \pi : X \rightarrow U)|_V := (\pi^{-1}(V), \omega|_{\pi^{-1}(V)}, \mu|_{\pi^{-1}(V)}).$$

Since morphisms in each groupoid  $\text{STSS}_\psi(U)$  must preserve  $G$ -quotients, the restriction of a morphism  $f : (X, \omega, \pi : X \rightarrow U) \rightarrow (X', \omega', \pi' : X' \rightarrow U')$  descends to a unique morphism in  $\text{STSS}_\psi(V)$ , giving a well-defined choice for  $f|_V$ . It is easy to check that these restriction maps satisfy the necessary requirements for a presheaf.

Implicit here is the fact that, for  $U$  not containing singularities, the condition that an object of  $\text{STSS}_\psi(U)$  must be modeled on certain neighborhoods of singular symplectic toric cones is empty. Hence, here the presheaves  $\text{STSS}_\psi$  and  $\text{STM}_{\psi|_{W_{\text{reg}}}}$  (see Definition 2.4) agree.

The restriction functor  $\rho_{W|W_{\text{reg}}} : \text{STSS}_\psi(W) \rightarrow \text{STSS}_\psi(W_{\text{reg}}) = \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$ , which from here forward we will denote  $\text{res}$ , will be important for us. We provide a specific definition of  $\text{res}$  now for completeness:

**Definition 9.6.** Given a stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , let

$$\text{res} : \text{STSS}_\psi(W) \rightarrow \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$$

be the functor taking a symplectic toric stratified space  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  to the symplectic toric manifold  $(X_{\text{reg}}, \omega, \mu|_{X_{\text{reg}}})$  and a morphism of symplectic toric stratified spaces  $\varphi : (X, \omega, \mu) \rightarrow (X', \omega', \mu')$  to the restriction  $\varphi|_{X_{\text{reg}}} : (X_{\text{reg}}, \omega, \mu|_{X_{\text{reg}}}) \rightarrow (X'_{\text{reg}}, \omega', \mu'|_{X'_{\text{reg}}})$  (which is, by definition, a moment map preserving symplectomorphism).

**Lemma 9.7.** For any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , the functor  $\text{res} : \text{STSS}_\psi(W) \rightarrow \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  is fully faithful.

*Proof.* Let's start by showing  $\text{res}$  is faithful. Given any two symplectic toric stratified spaces over  $\psi$   $(X, \omega, \pi : X \rightarrow W)$  and  $(X', \omega', \pi' : X' \rightarrow W)$ , for each singularity  $x_\alpha$  of  $X$ , it must be the case that any map  $\varphi$  of symplectic toric stratified spaces over  $\psi$  must satisfy  $\varphi(x_\alpha) = \pi'^{-1}(\pi(x_\alpha))$ . Since the singularities of both  $X$  and  $X'$  are in bijection with those of  $W$  via their respective quotient maps, there is no question as to where  $\varphi$  must send each singularity.

For convenience, write  $x'_\alpha = \pi'^{-1}(\pi(x_\alpha))$ . Then, by the logic above, every  $\varphi$  must satisfy  $\varphi(x_\alpha) = x'_\alpha$ , so if a map of symplectic toric manifolds  $\phi : (X_{\text{reg}}, \omega, \mu|_{X_{\text{reg}}}) \rightarrow (X'_{\text{reg}}, \omega', \mu'|_{X'_{\text{reg}}})$  admits an extension to a map of symplectic toric stratified spaces, this extension must be unique. Thus,  $\text{res}$  is faithful.

To show it is full, it is enough to show the unique extension of  $\phi$  described above is continuous. This follows much as in the proof of Proposition 8.12. As a neighborhood of each singularity of  $X'$  is symplectomorphic to a neighborhood of  $-\infty$  of a symplectic cone, we have seen in the previously cited proposition that every neighborhood of the singularity may be written as the union of a  $G$ -invariant neighborhood of  $-\infty$   $U$  and some open subset  $V$  of  $X'_{\text{reg}}$ . Then  $\phi^{-1}(U \cup V) = \phi^{-1}(U) \cup \phi^{-1}(V)$ .  $\phi^{-1}(V)$  is an open subset of  $X_{\text{reg}}$  which we will now ignore.

It remains to show  $\phi^{-1}(U)$  is a (deleted) open neighborhood of  $x_\alpha$  in  $X$ . As  $\phi$  is  $G$ -equivariant and  $U$  is  $G$ -invariant,  $\phi^{-1}(U)$  is also  $G$ -invariant. Thus,  $\phi^{-1}(U) = \pi^{-1}(\pi'(\pi^{-1}(U)))$ . So, since  $\pi'(U \sqcup \{x'_\alpha\})$  is an open neighborhood of  $\pi'(x'_\alpha)$  in  $W$ , it follows  $\phi^{-1}(U \sqcup \{x'_\alpha\})$  is an open neighborhood of  $x_\alpha$  in  $X$ . Thus, the extension of  $\phi$  is continuous. It follows that the extension of  $\phi^{-1}$  also continuous and, therefore,  $\phi$  extends to a homeomorphism between  $X$  and  $X'$  covering the identity on  $W$ . So  $\text{res}$  is full.  $\square$

To finish this section, we describe how symplectic toric stratified spaces may be pulled back over certain open embeddings of manifolds with corners.

**Lemma 9.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  and  $\psi' : W' \rightarrow \mathfrak{g}^*$  be two stratified unimodular local embeddings and let  $\varphi : W' \rightarrow W$  be an open embedding of cornered stratified spaces with  $\psi \circ \varphi = \psi'$ . Then, for any  $(X, \omega, \pi : X \rightarrow W) \in \text{STSS}_\psi(W)$ , there exists a stratified symplectic toric space  $(X', \omega', \pi' : X' \rightarrow W')$  and a  $G$ -equivariant isomorphism of symplectic stratified spaces  $\tilde{\varphi} : (X', \omega') \rightarrow (X, \omega)|_{\varphi(W')}$  with  $\pi \circ \tilde{\varphi} = \varphi \circ \pi'$ . We denote  $(X', \omega', \pi' : X' \rightarrow W')$  by  $\varphi^*(X, \omega, \pi : X \rightarrow W)$ .

*Proof.* Let  $\varphi(W') := U$ . As a stratified symplectic  $G$ -space, we simply take  $(X', \omega') = (X, \omega)|_U$ . Since  $\varphi$  is an open embedding, we have a homeomorphism  $\varphi^{-1} : U \rightarrow W'$  and so the map  $\pi' := \varphi^{-1}|_U \circ \pi : X' \rightarrow W'$  is a  $G$ -quotient for  $X'$ . Since  $(X, \omega)|_U$  has moment map

$$\psi \circ \pi = \psi \circ \varphi \circ \varphi^{-1} \circ \pi = \psi' \circ \pi',$$

it follows that  $(X', \omega', \pi' : X' \rightarrow W')$  is a symplectic toric stratified space over  $\psi' : W' \rightarrow \mathfrak{g}^*$ . By design,  $\tilde{\varphi}$  is simply the embedding of  $X' = \pi^{-1}(U)$  into  $X$ .  $\square$

We can also pullback isomorphisms.

**Lemma 9.9.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  and  $\psi' : W' \rightarrow \mathfrak{g}^*$  be two stratified unimodular local embeddings and let  $\varphi : W' \rightarrow W$  be an open embedding of cornered stratified spaces with  $\psi \circ \varphi = \psi'$ . Then, for any  $(X, \omega, \pi : X \rightarrow W)$  and  $(X', \omega', \pi' : X' \rightarrow W)$  in  $\text{STSS}_\psi(W)$  and isomorphism

$$f : (X, \omega, \pi : X \rightarrow W) \rightarrow (X', \omega', \pi' : X' \rightarrow W)$$

(also in  $\text{STSS}_{\psi'}(W')$ ) we may also pullback  $f$ ; that is, there is an isomorphism

$$\varphi^* f : \varphi^*(X, \omega, \pi : X \rightarrow W) \rightarrow \varphi^*(X', \omega', \pi' : X' \rightarrow W)$$

in  $\text{STSS}_{\psi'}(W')$ .

*Proof.* Let  $\tilde{\varphi} : \varphi^*(X, \omega) \rightarrow (X, \omega)|_{\varphi(W')}$  and  $\tilde{\varphi}' : \varphi^*(X', \omega') \rightarrow (X', \omega')|_{\varphi(W')}$  be the isomorphisms of Lemma 9.8. It is easy then to check that  $\varphi^* f := \tilde{\varphi}'^{-1} \circ f|_{\varphi(W')} \circ \tilde{\varphi}$  is an isomorphism in  $\text{STSS}_{\psi'}(W')$  as required.  $\square$

## 10. CONICAL SYMPLECTIC TORIC BUNDLES

As in [13] and in Part I, we define a new groupoid of principal  $G$ -bundles with special properties. The bundles in this section, conical symplectic toric bundles, are symplectic toric bundles with an extra condition to satisfy over deleted open neighborhoods of the singularity of the base cornered stratified space.

**Definition 10.1.** Let  $(W, W_{\text{reg}} \sqcup_{\alpha \in I} \{w_\alpha\})$  be a cornered stratified space with isolated singularities and let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then a **conical symplectic toric principal  $G$ -bundle** is a symplectic toric bundle  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  over  $\psi|_{W_{\text{reg}}}$  (see Definition 2.5) satisfying the following local condition: for each singularity  $w_\alpha$  of  $W$ , there exists

- a neighborhood  $U$  of  $w_\alpha$  in  $W$  with a homogeneous local trivialization datum  $\varphi : U \rightarrow c(L)$  and homogeneous unimodular local embedding  $\phi : \times \mathbb{R} \rightarrow \mathfrak{g}^*$  (see Definition 9.2);
- a homogeneous symplectic toric bundle  $(\varpi : Q \rightarrow L \times \mathbb{R}, \eta)$ ; and
- writing  $V := \varphi(U)$ , there is a  $G$ -equivariant symplectomorphism  $\tilde{\varphi} : (P|_{U_{\text{reg}}}, \omega) \rightarrow (Q|_{V_{\text{reg}}}, \eta)$  so that the diagram

$$\begin{array}{ccc} P|_{U_{\text{reg}}} & \xrightarrow{\tilde{\varphi}} & Q|_{V_{\text{reg}}} \\ \pi \downarrow & & \downarrow \varpi \\ U & \xrightarrow{\varphi} & c(L) \end{array}$$

commutes.



The groupoid  $\text{CSTB}_\psi(W)$  of conical symplectic toric bundles over  $\psi$  is the groupoid with objects as described above and morphisms  $G$ -equivariant symplectomorphisms covering the identity on  $W_{\text{reg}}$ .

**Remark 10.2.** As in the case of  $\text{STSS}_\psi$  (see Remark 9.5), we have a presheaf of groupoids  $\text{CSTB}_\psi : \text{Open}(W)^{\text{op}} \rightarrow \text{Groupoids}$ . Open subsets  $U$  of  $W$  not containing singularities renders the extra conditions of Definition 10.1 empty and here  $\text{CSTB}_\psi(U)$  and  $\text{STB}_{\psi|_{W_{\text{reg}}}}(U)$  (the category of symplectic toric bundles over  $U$ , see Definition 2.5) agree. This is the content of the functor  $\iota$  and the lemma below.

In fact,  $\text{CSTB}_\psi$  is a stack over  $W$ . As the proof of this is more or less just a retelling of the proof that the presheaf of principal bundles over a topological space is a stack, we relegate this proof to the appendix (Proposition B.9).

As in the case of symplectic toric stratified spaces (see Lemma 9.8), conical symplectic toric bundles may be pulled back over open embeddings of conical stratified spaces that preserve the respective stratified unimodular local embeddings.

**Lemma 10.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  and  $\psi' : W' \rightarrow \mathfrak{g}^*$  be two stratified unimodular local embeddings and let  $\varphi : W' \rightarrow W$  be an open embedding of cornered stratified spaces with  $\psi' \circ \varphi = \psi$ . Then, for  $(\pi : P \rightarrow W_{\text{reg}}, \omega) \in \text{CSTB}_\psi(W)$ , there exists a conical symplectic toric bundle  $(\pi' : P' \rightarrow W', \omega')$  and a  $G$ -equivariant symplectomorphism

$$\tilde{\varphi} : (\pi' : P' \rightarrow W', \omega') \rightarrow (\pi : P \rightarrow W_{\text{reg}}, \omega)|_{\varphi(W')}$$

with  $\pi \circ \tilde{\varphi} = \varphi \circ \pi'$ . We denote  $(P', \omega')$  as  $\varphi^*(P, \omega)$ .

*Proof.* As a symplectic  $G$ -space, let  $(P', \omega') = (P, \omega)|_{\varphi(W')}$ . Since  $\varphi : W' \rightarrow W$  is an open embedding, there is an inverse homeomorphism  $\varphi^{-1} : \varphi(W') \rightarrow W'$ . Let  $\pi' : P' \rightarrow W'$  be the map  $\varphi^{-1} \circ \pi$ . Then clearly  $(\pi' : P' \rightarrow W', \omega')$  is a conical symplectic toric bundle over  $\psi'$  and the embedding from  $P' = P|_{\varphi(W')}$  to  $P$  satisfies the requirements for  $\tilde{\varphi}$ .  $\square$

Pullbacks allow us to more efficiently describe conical symplectic toric bundles. We first need the following observation regarding shifted moment maps.

**Remark 10.4.** Given a unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  and a Lie algebra dual element  $\eta \in \mathfrak{g}^*$ , let  $\psi'$  be the map  $\psi'(w) := \psi(w) + \eta$ . Then  $\psi'$  is also a unimodular local embedding. Let  $(\pi : P \rightarrow W, \omega)$  be a symplectic toric bundle over  $\psi$ . Then clearly  $(\pi : P \rightarrow W, \omega)$  is a symplectic toric bundle over  $\psi'$  as well.

For  $C_w$  and  $C'_w$  the unimodular cones for  $w$ , defined relative to  $\psi$  and  $\psi'$  respectively, it is more or less obvious that  $C_w + \eta = C'_w$  and therefore both  $\psi$  and  $\psi'$  determine the same subtorus  $K_w$ . It follows that  $c_{\text{Top}}((P, \omega))$  is the same topological  $G$ -space when considering  $(P, \omega)$  as a symplectic toric bundle over  $\psi$  or  $\psi'$ . Following Construction 2.8, it is clear that, since the cutting procedures with respect to  $\psi$  or  $\psi'$  are performed relative to the cone basepoints  $\psi(w)$  and  $\psi'(w)$ ,  $c_{\text{Top}}((P, \omega))$  is symplectized the same way with respect to either unimodular local embedding.

Therefore, with respect to the identity map on  $c_{\text{Top}}((P, \omega))$ , the symplectic toric manifolds  $c((P, \omega)) \in \text{STM}_\psi(W)$  and  $c((P, \omega)) \in \text{STM}_{\psi'}(W)$  are symplectomorphic. Of course, this symplectomorphism does not preserve the respective moment maps.

Now, our lemma regarding pullbacks.

**Lemma 10.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then a symplectic toric bundle  $(\pi : P \rightarrow W_{\text{reg}})$  is a conical symplectic toric bundle exactly when, for each singularity  $w_\alpha$  of  $W$ , there exists

- an open neighborhood  $U$  of  $w_\alpha$ ;
- a homogeneous local trivialization datum  $\varphi : U \rightarrow c(L)$  with homogeneous unimodular local embedding  $\phi : L \times \mathbb{R} \rightarrow \mathfrak{g}^*$  satisfying  $\psi|_{U_{\text{reg}}} = \phi \circ \varphi + \psi(w)$ ; and
- a homogeneous symplectic toric bundle  $(\varpi : Q \rightarrow L \times \mathbb{R}, \eta) \in \text{HSTB}_\phi(L \times \mathbb{R})$

so that, thinking of  $(Q, \eta)$  as a symplectic toric bundle over  $\varphi + \psi(w_\alpha)$  (see Remark 10.4),  $\varphi^*(Q, \eta)$  and  $(P, \omega)|_U$  are isomorphic in  $\text{CSTB}(U)$ .

*Proof.* This is easily confirmed from the definition of a conical symplectic toric bundle (see Definition 10.1) and the description of a pullback of a conical symplectic toric bundle (see Lemma 10.3).  $\square$

We now discuss an important fully faithful functor  $\iota$ .

**Definition 10.6.** Given a stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , let  $\iota : \text{CSTB}_\psi(W) \rightarrow \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  denote the forgetful functor. Explicitly, as an object  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  in  $\text{CSTB}_\psi(W)$  is a symplectic toric  $G$ -bundle over  $W_{\text{reg}}$  (with an extra conical condition in neighborhoods of the singularities of  $W$ ), we may naturally define  $\iota(\pi : P \rightarrow W_{\text{reg}}, \omega) := (\pi : P \rightarrow W_{\text{reg}}, \omega)$  and, as a morphism in  $\text{CSTB}_\psi(W)$  is a symplectic isomorphism of principal  $G$ -bundles, any morphism in  $\text{CSTB}_\psi(W)$  descends to a morphism  $\iota(\varphi)$  in  $\text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$ .

**Lemma 10.7.** For any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ ,  $\iota : \text{CSTB}_\psi(W) \rightarrow \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  is fully faithful.

*Proof.* As explained in Remark 10.2,  $\text{CSTB}_\psi(W_{\text{reg}})$  and  $\text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  agree. More concretely, both are exactly the groupoid of symplectic toric bundles over the unimodular local embedding  $\psi|_{W_{\text{reg}}} : W_{\text{reg}} \rightarrow \mathfrak{g}^*$ . With this perspective,  $\iota$  is just the restriction functor in the presheaf  $\text{CSTB}_\psi$  taking elements of  $\text{CSTB}_\psi(W)$  to  $\text{CSTB}_\psi(W_{\text{reg}})$ . That  $\iota$  is fully faithful is more or less obvious from its definition.  $\square$

It is not clear that  $\text{CSTB}_\psi(W)$  is non-empty for certain choices of  $W$  or  $\psi$ . Again, as in [13], we can use a connection 1-form on any principal bundle to create a symplectic form with respect to which the corresponding bundle is conical symplectic toric. We will need the following fact about connections on principal bundles.

**Lemma 10.8.** Let  $\pi : P \rightarrow B$  be a principal  $K$ -bundle with connection 1-form  $A$ , where  $K$  is a commutative Lie group. Suppose also that  $X$  is any element of  $K$ 's Lie algebra  $\mathfrak{k}$ . Then  $d\langle A, X \rangle$  is a basic 2-form.

*Proof.* As  $K$  is commutative, note that  $dA$  is exactly the curvature of  $A$ . Furthermore, it is a standard fact (see, for instance, Proposition 6.39, pp. 266, [21]) that  $dA$  is:

- (1) Horizontal:  $dA|_{\ker \pi} = 0$
- (2) Equivariant:  $\rho_k^* dA = \text{Ad}_{k^{-1}} \circ dA$  for any  $k \in K$

Again, as  $K$  is commutative,  $\text{Ad}_k = \text{id}_{\mathfrak{k}}$  for any  $k \in K$ , so the second condition actually implies that  $dA$  is  $K$ -invariant. It follows that  $dA$  is basic (again, this is standard; see Lemma 6.44, pp. 275, [21]). Therefore, the form  $d\langle A, X \rangle = \langle dA, X \rangle$  is basic.

While the above references are for manifolds only, it is easy to show that they must also apply to principal  $K$ -bundles of manifolds with corners.  $\square$

**Proposition 10.9.** For  $\psi : W \rightarrow \mathfrak{g}^*$  a stratified unimodular local embedding and any principal bundle  $\pi : P \rightarrow W_{\text{reg}}$ , there exists an exact  $G$ -invariant symplectic form  $\omega$  on  $P$  so that  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is a conical symplectic toric bundle over  $\psi$ .

*Proof.* Recall from Definition 9.2, for each singularity  $w_\alpha$  in  $W$ , there exists an open subset  $U_\alpha$  of  $w_\alpha$  in  $W$ , local trivialization datum  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$ , and a homogeneous unimodular local embedding  $\phi_\alpha : L_\alpha \times \mathbb{R} \rightarrow \mathfrak{g}^*$  such that  $\psi|_{U_\alpha} = \phi_\alpha \circ \varphi_\alpha + \psi(w_\alpha)$ . Fix such a piece of data for each  $\alpha$ . Complete the set  $\{U_\alpha\}_{\alpha \in A}$  to an open cover of all  $W$  with an open set  $U_0 \subset W_{\text{reg}}$  so that, for each  $\alpha$ , there is an open neighborhood of  $w_\alpha$   $U'_\alpha \subset U_\alpha$  with  $U'_\alpha \cap U_0$  empty. Then, for each  $\alpha$ , we will build a  $G$ -invariant symplectic form for  $P|_{U_{\alpha\text{reg}}}$  and show these forms may be patched together to a form with the properties we desire.

Our key tool will be Proposition 4.6. Without loss of generality, we may assume that for each  $\alpha$ ,  $\varphi_\alpha : U_\alpha \rightarrow c(L_\alpha)$  is the inclusion  $\iota : L_\alpha \times (-\infty, \epsilon) \sqcup \{*\} \rightarrow c(L)$  for some real number  $\epsilon$ . Fix a number  $\tau < \epsilon$  and let  $B := L_\alpha \times \{\tau\}$ . Then  $\varpi : Q \rightarrow B$  is a principal  $G$ -bundle as well. Note that the projection  $p : L_\alpha \times (-\infty, \epsilon) \rightarrow L_\alpha \times (-\infty, \epsilon)$  of  $L_\alpha \times (-\infty, \epsilon)$  onto  $B$  is homotopy equivalent to the identity on  $L_\alpha \times (-\infty, \epsilon)$ .

Therefore, by Theorem 6.4, the bundles  $p^*(P|_{L_\alpha \times (-\infty, \epsilon)})$  and  $P|_{L_\alpha \times (-\infty, \epsilon)}$  are isomorphic via some isomorphism of principal  $G$ -bundles  $f : P|_{L_\alpha \times (-\infty, \epsilon)} \rightarrow p^*(P|_{L_\alpha \times (-\infty, \epsilon)})$ . Since we may represent the pullback  $p^*(P|_{L_\alpha \times (-\infty, \epsilon)})$  by the bundle  $(\varpi \times id) : Q \times (-\infty, \epsilon) \rightarrow L_\alpha \times (-\infty, \epsilon)$  (with an implicit identification of  $B$  with  $L_\alpha$ ), we will take the image of  $f$  to be this bundle.

Now, note that we may extend  $Q \times (-\infty, \epsilon)$  to the principal  $G$ -bundle  $(\varpi \times id) : L_\alpha \times \mathbb{R} \rightarrow B \times \mathbb{R}$  and that  $\varpi \times id$  is clearly  $\mathbb{R}$ -equivariant. Therefore, by Proposition 4.6, there exists a connection 1-form  $A'$  on  $Q \times \mathbb{R}$  so that  $(Q \times \mathbb{R}, d\langle A', \phi_\alpha \circ (\varpi \times id) \rangle)$  is a homogeneous symplectic toric bundle over  $\phi_\alpha$ .

Write  $A_\alpha := f^*(A'|_{Q \times (-\infty, \epsilon)})$ . Then, for  $\iota' : Q \times (-\infty, \epsilon) \rightarrow Q \times \mathbb{R}$  the inclusion, we have:

$$\begin{aligned} f^* \iota'^*(d\langle A', \phi_\alpha \circ (\varpi \times id) \rangle) &= f^*(d\langle A'|_{Q \times (-\infty, \epsilon)}, \phi_\alpha \circ \iota \circ (\varpi \times id) \rangle) \\ &= d\langle A_\alpha, \phi_\alpha \circ \iota \circ \pi|_{U_{\alpha\text{reg}}} \rangle \\ &= d\langle A_\alpha, \psi \circ \pi|_{U_{\alpha\text{reg}}} - \psi(w_\alpha) \rangle \end{aligned}$$

By Lemma 10.8,  $-\langle A_\alpha, \psi(w_\alpha) \rangle$  is basic. Let  $\gamma_\alpha$  be a form on  $U_{\alpha\text{reg}}$  with

$$\pi|_{U_{\alpha\text{reg}}}^* \gamma_\alpha := -\langle A_\alpha, \psi(w_\alpha) \rangle.$$

To finish, let  $\{A_\alpha\}_{\alpha \in A}$  be the collection of connection 1-forms on each  $P|_{U_{\alpha\text{reg}}}$  selected as above and let  $A_0$  be any connection 1-form for  $U_0$  (for  $U_0$  as in the beginning of this proof). Let  $\{\{\rho_\alpha\}_{\alpha \in A}, \rho_0\}$  be a partition of unity for the open cover  $\{\{U_{\alpha\text{reg}}\}_{\alpha \in A}, U_0\}$  of  $W_{\text{reg}}$ . Then  $A := \rho_0 A_0 + \sum_\alpha \rho_\alpha A_\alpha$  is a connection 1-form for  $P$ .

Define  $\gamma := \sum_\alpha \rho_\alpha \gamma_\alpha$ . Then by Lemma 2.12, the form  $\omega := d\langle A, \psi \circ \pi \rangle + d\pi^* \gamma$  is a  $G$ -invariant symplectic form on  $P$  with moment map  $\psi \circ \pi$ . it follows by our work above that  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is a conical symplectic toric bundle over  $\psi$ .  $\square$

## 11. THE MORPHISM OF PRESHEAVES $\tilde{c} : \text{CSTB}_\psi \rightarrow \text{STSS}_\psi$

In this section, we will define the functor  $\tilde{c} : \text{CSTB}_\psi(W) \rightarrow \text{STSS}_\psi(W)$  for any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ . This is done using the following steps.

**Step 1:** We must first define the functor  $\tilde{c}_{\text{top}}$  taking conical symplectic toric bundles  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  in  $\text{CSTB}_\psi(W)$  to pairs of topological  $G$ -spaces with quotient maps to  $W$   $\tilde{c}_{\text{top}}(\pi : P \rightarrow W_{\text{reg}}, \omega) = (\bar{P}, \bar{\pi} : \bar{P} \rightarrow W)$  and maps of conical symplectic toric bundles to maps of topological  $G$ -spaces over  $W$ :  $G$ -equivariant homeomorphisms  $f : (X, \varpi : X \rightarrow W) \rightarrow (X', \varpi' : X' \rightarrow W)$  with  $\varpi = f \circ \varpi'$ .

**Step 2:** We may now define  $\tilde{c}((\pi : P \rightarrow W_{\text{reg}}, \omega))$ , a tuple

$$(X, X_{\text{reg}} \sqcup_{\alpha \in A} \{x_\alpha\}, \omega, \bar{\pi} : X \rightarrow W)$$

where  $(X, \bar{\pi}) = \tilde{c}_{\text{top}}(\pi : P \rightarrow W_{\text{reg}}, \omega)$  and  $X_{\text{reg}} \sqcup_{\alpha \in A} \{x_\alpha\}$  is a partition of  $X$  for which  $(X_{\text{reg}}, \omega|_{X_{\text{reg}}} : X_{\text{reg}} \rightarrow W_{\text{reg}})$  is a symplectic toric manifold over  $\psi|_{W_{\text{reg}}} : W_{\text{reg}} \rightarrow \mathfrak{g}^*$  (we will call this type of object a *partitioned symplectic toric space over  $\psi$* ). We may also show that, for any morphism of conical symplectic toric principal bundles  $\varphi$ , the morphism  $\tilde{c}(\varphi) = \tilde{c}_{\text{top}}(\varphi)$  restricts to a symplectomorphism on the open dense pieces of the source and target partitioned symplectic toric spaces.

**Step 3:** We next show  $\tilde{c}$  commutes with pullbacks: for any open embedding  $\varphi : W' \rightarrow W$  and stratified unimodular local embeddings  $\psi : W \rightarrow \mathfrak{g}^*$  and  $\psi' : W' \rightarrow \mathfrak{g}^*$  for which  $\psi \circ \varphi = \psi'$ , given any conical symplectic toric bundle  $(P, \omega)$ , if  $\tilde{c}(P, \omega)$  is a symplectic toric stratified space, then  $\tilde{c}(\varphi^*(P, \omega)) = \varphi^* \tilde{c}(P, \omega)$  (see Lemma 9.8 and Lemma 10.3 for the definitions of these pullbacks). In particular,  $\tilde{c}$  commutes with restrictions.

**Step 4:** We now show that, for a particular model case of  $\psi$ ,  $\tilde{c}$  takes certain conical symplectic toric bundles to singular symplectic toric cones.

**Step 5:** With a small amount of work, this allows us to show  $\tilde{c}(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is actually a symplectic toric stratified space over  $\psi$ .

We now flesh out the details of each step.

**Step 1:** To start, fix a stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$  and fix  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  a conical symplectic toric bundle in  $\text{CSTB}_\psi(W)$ . Suppose  $W$  has partition  $W = W_{\text{reg}} \sqcup_{\alpha \in A} \{w_\alpha\}$ . First we aim to define  $\tilde{c}_{\text{top}}(P, \omega)$ , a topological  $G$ -space with a  $G$ -quotient to  $W$ .

We construct the cornered stratified space  $\tilde{P}$  as follows: as a set,  $\tilde{P} := P \sqcup_{\alpha \in A} \{p_\alpha\}$  for a set of points  $\{p_\alpha\}$  in bijection with the singularities of  $W$ .

We then give  $\tilde{P}$  the topology generated by sets of the form:

- (1) open subsets of  $P$
- (2) sets  $\{p_\alpha\} \sqcup P|_{U_{\text{reg}}}$ , for  $U$  a neighborhood of  $w_\alpha$  in  $W$

Recall that, for each  $w \in W_{\text{reg}}$ , as  $\psi|_{W_{\text{reg}}}$  is a unimodular local embedding, there exists a subtorus  $K_w < G$  determined by the image of  $\psi$  in  $\mathfrak{g}^*$  (see Section 2). Then let  $\sim$  be the equivalence relation on  $\tilde{P}$  defined for elements of  $P \subset \tilde{P}$  by

$$p \sim p' \text{ when there exists } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'$$

and extended to  $\tilde{P}$  so that the added singularities  $p_\alpha$  occupy their own equivalence classes. Essentially, this is the same construction as  $c_{\text{top}}$  of [13], with the inclusion of the additional information of the singularities added to  $\tilde{P}$ .

Note that  $\tilde{P}$  inherits a  $G$  action which is the original  $G$  action on  $P$  extended trivially to  $\tilde{P}$  (i.e., so that each  $p_\alpha$  is fixed). It is clear that this is a continuous action on  $\tilde{P}$  since

any open set containing a singularity of  $p_\alpha$  must contain a set of the form  $P|_{U_{\text{reg}}} \sqcup \{p_\alpha\}$  in  $\tilde{P}$  which the action of any element of  $G$  must take to itself.

Therefore, the  $G$ -quotient  $\pi : P \rightarrow W_{\text{reg}}$  extends to a  $G$ -quotient  $\tilde{\pi} : \tilde{P} \rightarrow W$  with  $\tilde{\pi}(p_\alpha) = w_\alpha$  for every  $\alpha$ .

Now, let  $q : \tilde{P} \rightarrow \tilde{P}/\sim$  be the topological quotient map. Then, since  $\sim$  only identifies elements of the same  $G$ -orbit,  $\tilde{P}/\sim$  inherits a  $G$  action with respect to which  $q$  is equivariant; Namely, the action  $g \cdot [p] := [g \cdot p]$ . It is clear from the universal property of quotients that there exists a unique map  $\bar{\pi} : \tilde{P}/\sim \rightarrow W$  with  $\bar{\pi} \circ q = \tilde{\pi}$ . Furthermore, it is clear from how we've defined the  $G$  action on  $\tilde{P}/\sim$  that  $\bar{\pi}$  is the  $G$ -quotient map for this action. So define  $\tilde{c}_{\text{top}}(P, \omega) := (\tilde{P}, \bar{\pi} : \tilde{P} \rightarrow W)$ .

For  $\varphi : (P, \omega) \rightarrow (P', \omega')$  a map of conical symplectic toric bundles, note first that  $\varphi$  extends to a homeomorphism  $\tilde{\varphi}$  between  $\tilde{P} = P \sqcup_{\alpha \in A} \{p_\alpha\}$  and  $\tilde{P}' = P' \sqcup_{\alpha \in A} \{p'_\alpha\}$ . This follows from the fact that  $\varphi$  is a map of  $G$ -bundles and so takes an open set of the form  $P|_{U \cap W_{\text{reg}}}$  in  $P$  to the set  $P'|_{U \cap W_{\text{reg}}}$  in  $P'$ ; thus,  $\varphi$  takes open neighborhoods of  $p_\alpha$  to open neighborhoods of  $p'_\alpha$  and, similarly,  $\varphi^{-1}$  takes open neighborhoods of  $p'_\alpha$  to open neighborhoods of  $p_\alpha$ . Clearly, we also have that  $\tilde{\pi} = \tilde{\pi}' \circ \tilde{\varphi}$ .

So, since  $\tilde{\varphi} : \tilde{P} \rightarrow \tilde{P}'$  is a  $G$ -equivariant homeomorphism, it descends to a map

$$\tilde{c}_{\text{top}}(\varphi) : \tilde{P}/\sim \rightarrow \tilde{P}'/\sim \quad \tilde{c}_{\text{top}}(\varphi)([p]) := [\varphi(p)]. \quad (11.1)$$

It is clear from the way  $\tilde{\pi}$  and  $\tilde{\pi}'$  were defined that, since  $\tilde{\pi} = \tilde{\pi}' \circ \tilde{\varphi}$ , we must have  $\tilde{\pi} = \tilde{\pi}' \circ \tilde{c}_{\text{top}}(\varphi)$ . Therefore,  $\tilde{c}_{\text{top}}(\varphi)$  is a map of topological  $G$ -spaces over  $W$ .

**Step 2:** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding and let  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  be a conical symplectic toric bundle. Let  $\tilde{P}$  and  $\sim$  be the extension of  $P$  and equivalence relation as described in the previous step. It is clear from how we've defined  $\tilde{c}_{\text{top}}$  that the subset  $P \subset \tilde{P}$  descends to the open dense subset  $P/\sim \subset \tilde{P}$ . It is also clear from how  $c_{\text{Top}}$  was defined in [13] (see Section 2) that  $P/\sim$  is exactly the topological space  $c_{\text{Top}}(\iota(P, \omega))$  (recall  $\iota : \text{CSTB}_\psi(W) \rightarrow \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  is the restriction from  $W$  to  $W_{\text{reg}}$  in the presheaf  $\text{CSTB}_\psi$  together with the identification  $\text{CSTB}_\psi(W_{\text{reg}}) = \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$ ; see Remark 10.2 and Definition 10.6).

So  $\tilde{P}$  has partition  $c_{\text{Top}}(\iota(P, \omega)) \sqcup_{\alpha \in A} \{p_\alpha\}$ . Recall that, by definition,  $c(\iota(P, \omega))$  is a symplectic toric manifold over  $\psi|_{W_{\text{reg}}}$  homeomorphic to the topological space  $c_{\text{Top}}(P, \omega)$ . Write  $c(\iota(P, \omega)) = (M, \bar{\omega}, \varpi : M \rightarrow W_{\text{reg}})$ . The quotient map  $\varpi : M \rightarrow W_{\text{reg}}$  is just the  $G$ -quotient map for the topological space  $c_{\text{Top}}(\iota(P, \omega)) \subset \tilde{P}/\sim$  which is just the restriction of  $\tilde{\pi} : \tilde{P}/\sim \rightarrow W$ .

We now define:

$$\tilde{c}((\pi : P \rightarrow W_{\text{reg}}, \omega)) = (\tilde{P}/\sim, P/\sim \sqcup \{[p_\alpha]\}, \bar{\omega}, \tilde{\pi} : \tilde{P}/\sim \rightarrow W) \quad (11.2)$$

where  $P/\sim \sqcup \{[p_\alpha]\}$  is a partition for  $\tilde{P}$  and  $(P/\sim, \bar{\omega}, \tilde{\pi}|_{P/\sim})$  is a symplectic toric manifold over  $\psi|_{W_{\text{reg}}}$ . We will show that this is a symplectic toric stratified space but, for now, we will refer to this tuple as a *partitioned symplectic toric space over  $\psi$* . For notational convenience, we will often leave the partition as implicit.

Finally, let  $\varphi : (P, \omega) \rightarrow (P', \omega')$  be a map of conical symplectic toric bundles over  $\psi$ . Then since our definition for  $\tilde{c}_{\text{top}}(\varphi)$  matches  $c_{\text{Top}}(\iota(\varphi))$  (again, see Section 2), it follows that the morphism  $\tilde{c}_{\text{top}}(\varphi)$  restricts to an isomorphism of symplectic toric manifolds between the open

dense pieces of  $\tilde{\mathfrak{c}}(P, \omega)$  and  $\tilde{\mathfrak{c}}(P', \omega')$ . In the case where  $\tilde{\mathfrak{c}}(P, \omega)$  and  $\tilde{\mathfrak{c}}(P', \omega')$  are actually symplectic toric stratified spaces over  $\psi$ , this means exactly that  $\tilde{\mathfrak{c}}(\varphi)$  is an isomorphism of symplectic toric stratified spaces over  $\psi$ .

**Step 3:** The content of this step is the following two lemmas regarding  $\tilde{\mathfrak{c}}$  commuting with restrictions and, more generally, with pullbacks (as defined in Lemmas 9.8 and 10.3).

**Lemma 11.1.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding and let  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  be a conical symplectic toric manifold over  $\psi$ . Then for any open subset  $U$  in  $W$ , if  $\tilde{\mathfrak{c}}(P, \omega)$  is a symplectic toric stratified space, then so is  $\tilde{\mathfrak{c}}((P, \omega)|_U)$  and  $\tilde{\mathfrak{c}}((P, \omega)|_U) = \tilde{\mathfrak{c}}(P, \omega)|_U$ .

*Proof.* Fix open  $U$  in  $W$ . To prove both statements of the lemma, it is enough to show that the partitioned symplectic toric space  $\tilde{\mathfrak{c}}((P, \omega)|_U)$  is equal to the symplectic toric stratified space  $\tilde{\mathfrak{c}}(P, \omega)|_U$ . Note that the total space of  $(P, \omega)|_U$  is  $P|_{U_{\text{reg}}}$ . Recall from Step 2 that

$$\tilde{\mathfrak{c}}((P, \omega)) := (\tilde{P}/\sim, P/\sim \sqcup \{[p_\alpha]\}_{\alpha \in A}, \bar{\omega}, \bar{\pi} : \tilde{P}/\sim \rightarrow W).$$

Then for  $\tilde{\pi} : \tilde{P} \rightarrow W$  the  $G$ -quotient of  $\tilde{P}$  as in Step 1, it is easy to check from the definition of  $\tilde{\mathfrak{c}}_{\text{top}}$  that we can identify  $\widetilde{P|_{U_{\text{reg}}}}$  with the open subset  $\tilde{\pi}^{-1}(U)$  of  $\tilde{P}$ .

Let  $q : \tilde{P} \rightarrow \tilde{P}/\sim$  be the topological quotient map. Since  $\tilde{\pi}^{-1}(U)$  is  $G$ -invariant, the set  $q(\tilde{\pi}^{-1}(U))$  is an open neighborhood of  $\tilde{P}/\sim$ . Let  $\sim'$  be the relation defined for  $\widetilde{P|_{U_{\text{reg}}}}$  via  $\psi|_U : U \rightarrow \mathfrak{g}^*$ . As the equivalence relation  $\sim$  is defined via local data from  $\psi : W \rightarrow \mathfrak{g}^*$ , the relation  $\sim'$  on  $\widetilde{P|_{U_{\text{reg}}}}$  thought of as the subset  $\tilde{\pi}^{-1}(U)$  of  $\tilde{P}$  corresponds to  $\sim$ . Therefore, the quotient  $\widetilde{P|_{U_{\text{reg}}}}/\sim'$  may be identified with  $q(\tilde{\pi}^{-1}(U))$ .

By definition (again, see Step 1),  $\bar{\pi} \circ q = \tilde{\pi}$ , so  $\bar{\pi}(q(\tilde{\pi}^{-1}(U))) = U$ . Since  $q(\tilde{\pi}^{-1}(U))$  is  $G$ -invariant and  $\bar{\pi}$  is the  $G$ -quotient of  $\tilde{P}$ ,  $\bar{\pi}(q(\tilde{\pi}^{-1}(U))) = U$  implies that  $q(\tilde{\pi}^{-1}(U)) = \bar{\pi}^{-1}(U)$ . Therefore, as topological  $G$ -spaces over  $U$ ,  $\tilde{\mathfrak{c}}_{\text{top}}((P, \omega)|_U) = \tilde{\mathfrak{c}}_{\text{top}}((P, \omega))|_U$  (that is, the topological spaces and  $G$ -quotients of  $\tilde{\mathfrak{c}}((P, \omega)|_U)$  and  $\tilde{\mathfrak{c}}((P, \omega))|_U$  match).

Finally, recall we have a fully faithful functor  $\iota : \text{CSTB}_\psi(W) \rightarrow \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  described in Definition 10.6. It is clear, since this is just a forgetful functor (with an obvious identification of groupoids) that  $\iota((P, \omega)|_U) = \iota(P, \omega)|_{U_{\text{reg}}}$ . As  $c$  is a map of presheaves, we have  $c(\iota((P, \omega)|_{U_{\text{reg}}})) = c(\iota(P, \omega)|_{U_{\text{reg}}}) = c(\iota(P, \omega))|_{U_{\text{reg}}}$ . Therefore, the symplectic structure on the open dense pieces of  $\tilde{\mathfrak{c}}((P, \omega)|_U)$  and  $\tilde{\mathfrak{c}}((P, \omega))|_U$  match and  $\tilde{\mathfrak{c}}((P, \omega)|_U)$  and  $\tilde{\mathfrak{c}}((P, \omega))|_U$  are the same partitioned symplectic toric space; hence, under the assumption that  $\tilde{\mathfrak{c}}(P, \omega)$  is a symplectic toric stratified space, both partitioned symplectic toric spaces are in fact symplectic toric stratified spaces.  $\square$

**Lemma 11.2.** Suppose  $\varphi : W' \rightarrow W$  is an open embedding of cornered stratified spaces and  $\psi : W \rightarrow \mathfrak{g}^*$ ,  $\psi' : W' \rightarrow \mathfrak{g}^*$  are stratified unimodular local embeddings with  $\psi \circ \varphi = \psi'$ . Then for any conical symplectic toric bundle  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  for which  $\tilde{\mathfrak{c}}(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is a symplectic toric stratified space over  $\psi$ ,  $\tilde{\mathfrak{c}}(\varphi^*(\pi : P \rightarrow W_{\text{reg}}, \omega)) = \varphi^*\tilde{\mathfrak{c}}(\pi : P \rightarrow W_{\text{reg}}, \omega)$ .

*Proof.* As usual, we write  $\tilde{\mathfrak{c}}(P, \omega) = (\tilde{P}/\sim, \bar{\omega}, \bar{\pi} : \tilde{P}/\sim \rightarrow W)$ . Recall that the pullbacks are constructed, for  $U := \varphi(W') \subset W$ , as:

$$\varphi^*(\pi : P \rightarrow W_{\text{reg}}, \omega) = (\varphi^{-1} \circ \pi : P|_{U_{\text{reg}}} \rightarrow W'_{\text{reg}}, \omega)$$

and

$$\varphi^*\tilde{\mathfrak{c}}(P, \omega) = (\bar{\pi}^{-1}(U), \bar{\omega}, \varphi^{-1} \circ \bar{\pi} : \bar{\pi}^{-1}(U) \rightarrow W'_{\text{reg}}).$$

As shown in the proof of Lemma 11.1,  $\bar{\pi}^{-1}(U) = \widetilde{P|_{U_{\text{reg}}}} / \sim$ . Thus,

$$\check{c}(\varphi^*(P, \omega)) = (\bar{\pi}^{-1}(U), \bar{\omega}, \overline{\varphi^{-1} \circ \pi} : \bar{\pi}^{-1}(U) \rightarrow W')$$

for  $\overline{\varphi^{-1} \circ \pi}$  the quotient map defined as in Step 1. So it is enough to show that  $\overline{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \bar{\pi}$ .

Recall that, when constructing  $\overline{\varphi^{-1} \circ \pi}$ , we first build  $\widetilde{\varphi^{-1} \circ \pi} : \tilde{P} \rightarrow W'$ , the extension of  $\varphi^{-1} \circ \pi$  to  $\tilde{P}$ . It is clear that, for  $\tilde{\pi} : \tilde{P} \rightarrow W$  the extension of  $\pi$  to  $\tilde{P}$ , that  $\widetilde{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \tilde{\pi}$ .

Next, for  $q : \tilde{P} \rightarrow \tilde{P} / \sim$  the topological quotient map,  $\overline{\varphi^{-1} \circ \pi} : \tilde{P} / \sim \rightarrow W'$  is by definition the unique map for which  $\overline{\varphi^{-1} \circ \pi} \circ q = \widetilde{\varphi^{-1} \circ \pi}$ . But since  $\bar{\pi} \circ q = \tilde{\pi}$ , we have:

$$\overline{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \tilde{\pi} = \varphi^{-1} \circ \bar{\pi} \circ q$$

Therefore, by uniqueness from the universal property of a quotient,  $\overline{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \bar{\pi}$ .  $\square$

**Step 4:** We now prove that  $\check{c}(P, \omega)$  is a symplectic toric stratified space for a particular case of stratified unimodular local embedding.

**Proposition 11.3.** Let  $B$  be a manifold with corners and let  $\psi : c(B) \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding for which  $\psi|_{B \times \mathbb{R}}$  is a homogeneous unimodular local embedding with respect to the action of translation on the second factor (see Definition 3.2). Suppose that  $(\pi : P \rightarrow B \times \mathbb{R}, \omega)$  is a homogeneous symplectic toric bundle in  $\text{HSTB}_{\psi|_{B \times \mathbb{R}}}(B \times \mathbb{R})$ . Then  $(P, \omega)$  is a conical symplectic toric bundle over  $\psi$  and  $\check{c}(P, \omega)$  is a singular symplectic toric cone over  $\psi$ .

*Proof.* First, it is clear from the definition of a conical symplectic toric bundle that  $(P, \omega)$ , the homogeneous symplectic toric bundle over  $\psi|_{B \times \mathbb{R}}$ , is a conical symplectic toric bundle over  $\psi$ . As in the steps above, we have  $\check{c}(P, \omega) = (\tilde{P} / \sim, \bar{\omega}, \bar{\pi} : \tilde{P} / \sim \rightarrow B)$ .

Let  $\tilde{P}$  and  $\tilde{\pi} : \tilde{P} \rightarrow W$  correspond to the same extension of  $P$  and  $\pi$  in Step 1 when constructing  $\check{c}_{\text{top}}$ . Then it is apparent from how both spaces are defined that  $(\tilde{P}, \omega, \psi \circ \tilde{\pi})$  is the extension of the symplectic toric cone  $(P, \omega, \psi \circ \pi)$  to a singular symplectic toric cone as given in Proposition 8.12.

Since  $P \subset \tilde{P}$  is a homogeneous symplectic toric bundle over  $\psi|_{B \times \mathbb{R}}$ , we may apply  $\text{hc}$  to  $(P, \omega)$ . Recall that  $\text{hc} : \text{HSTB}_{\psi|_{B \times \mathbb{R}}}(B \times \mathbb{R}) \rightarrow \text{STC}_{\psi|_{B \times \mathbb{R}}}(B \times \mathbb{R})$  is just the functor  $c$  that remembers the action of  $\mathbb{R}$  on a homogeneous symplectic toric bundle (see Definition 5.2). Then by using  $\text{hc}$ ,  $c(P, \omega)$ , the symplectic toric manifold over  $\psi|_{B \times \mathbb{R}}$ , inherits the structure of a symplectic toric cone over  $\psi|_{B \times \mathbb{R}}$ .

To finish, let  $L := P / \mathbb{R}$  and let  $\varphi : P \rightarrow L \times \mathbb{R}$  be a  $G$ -equivariant trivialization of  $P$  as a principal  $\mathbb{R}$ -bundle (by Proposition A.10, such a trivialization exists). Then, by Proposition 8.7, this extends to a homeomorphism  $\tilde{\varphi} : \tilde{P} \rightarrow c(L)$ . Note that we may apply the construction  $\text{c}_{\text{Top}}$  to each slice  $L \times \{\tau\} \rightarrow B \times \mathbb{R}$  and, since the actions of  $G$  and  $\mathbb{R}$  on  $L \times \mathbb{R}$  commute, it follows that the topological  $G$ -spaces  $\text{c}_{\text{Top}}(L \times \{\tau\})$  are equivariantly homeomorphic for each  $\tau$ . So, defining  $L' := \text{c}_{\text{Top}}(L \times \{0\})$ , it follows that  $\check{c}_{\text{top}}(c(L)) \cong c(L')$  as topological  $G$ -spaces.

The map  $\varphi : P \rightarrow L \times \mathbb{R}$  then descends to a  $(G \times \mathbb{R})$ -equivariant homeomorphism  $\phi : \tilde{P} / \sim \rightarrow c(L')$ . As  $\phi$  restricts to a  $(G \times \mathbb{R})$ -equivariant homeomorphism between  $\text{hc}(P, \omega) \subset \tilde{P} / \sim$  and  $L' \times \mathbb{R}$ , we may conclude that  $L'$  is in fact diffeomorphic to the manifold  $\text{hc}(P, \omega) / \mathbb{R}$ .

Therefore,  $\phi$  gives local trivialization data for  $[*]$  in  $\tilde{P}$  and so  $(\tilde{P}/\sim, P/\sim \sqcup \{[*]\})$  is a stratified space with one singularity.

It follows by definition that  $\tilde{c}(P, \omega) = (\tilde{P}/\sim, P/\sim \sqcup \{[*]\}, \text{hc}(\omega), \tilde{\pi} : \tilde{P}/\sim \rightarrow c(B))$  is a singular symplectic toric cone over  $\psi$  (i.e., a singular symplectic toric cone with moment map  $\psi \circ \tilde{\pi}$ ).  $\square$

**Step 5:** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding and let  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  be a conical symplectic toric bundle. Let  $\tilde{P}, \tilde{\pi}, \sim$ , etc. be the same objects as defined in the previous steps. The purpose of this step is to show that the partitioned symplectic toric space over  $\psi$   $\tilde{c}((\pi : P \rightarrow W_{\text{reg}}, \omega))$  defined in Step 2 is in fact a symplectic toric stratified space. To do this, we must show that

- (1) With respect to its partition,  $\tilde{c}((\pi : P \rightarrow W_{\text{reg}}, \omega))$  is a stratified space with isolated singularities
- (2) Each singularity of  $\tilde{c}((\pi : P \rightarrow W_{\text{reg}}, \omega))$  has a neighborhood isomorphic to the neighborhood of the singularity in a singular symplectic toric cone (see Definition 9.1)

As neighborhoods of the singularity of any singular symplectic toric cone are clearly homeomorphic to  $c(L)$  for some compact manifold  $L$ , if we can show the second condition above holds, the first must hold as well, so it is enough to only address (2).

So fix a singularity  $p_\alpha$  of  $\tilde{P}$  lying over  $w_\alpha$  in  $W$ . Then by Lemma 10.5, there exists an open neighborhood  $U$  of  $w_\alpha$ , a homogeneous local trivialization datum  $\varphi : U \rightarrow c(L)$  with homogeneous unimodular local embedding  $\phi : L \times \mathbb{R} \rightarrow \mathfrak{g}^*$  satisfying  $\psi|_{U_{\text{reg}}} = \phi \circ \varphi + \psi(w)$ , a homogeneous symplectic toric bundle  $(\varpi : Q \rightarrow L \times \mathbb{R}, \eta) \in \text{HSTB}_\phi(L \times \mathbb{R})$ , and a  $G$ -equivariant isomorphism  $f : (P, \omega)|_U \rightarrow \varphi^*(Q, \eta)$  in  $\text{CSTB}_\psi(U)$ .

Then since  $\tilde{c}(\varphi^*(Q, \eta)) = \varphi^*\tilde{c}(Q, \eta)$  and  $\tilde{c}((P, \omega)|_U) = \tilde{c}(P, \omega)|_U$ , we have an isomorphism

$$\tilde{c}(f) : \tilde{c}(P, \omega)|_U \rightarrow \varphi^*\tilde{c}(Q, \eta).$$

Thus, we have an isomorphism from  $\tilde{c}(P, \omega)|_U$  to the neighborhood of the singularity  $\varphi^*\tilde{c}(Q, \eta)$  of the singular symplectic toric cone  $\tilde{c}(Q, \eta)$ .

We may now define  $\tilde{c}$ .

**Definition 11.4.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then  $\tilde{c} : \text{CSTB}_\psi(W) \rightarrow \text{STSS}_\psi(W)$  is the functor with  $\tilde{c}(P, \omega)$  the symplectic toric stratified space over  $\psi$  stated in equation (11.2) for  $(P, \omega) \in \text{CSTB}_\psi(W)$  and  $\tilde{c}(\varphi)$  the map stated in equation (11.1) for any morphism  $\varphi : (P, \omega) \rightarrow (P', \omega')$  in  $\text{CSTB}_\psi(W)$ . That this is well-defined is the content of the above steps.

**Remark 11.5.** As in the case of symplectic toric bundles and symplectic toric manifolds, we may also make the following observation about shifting moment maps. Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding, let  $\eta \in \mathfrak{g}^*$  and let  $\psi' : W \rightarrow \mathfrak{g}^*$  be the map  $\psi'(w) := \psi(w) + \eta$ . Then it is clear  $\psi'$  is also a stratified unimodular local embedding. Write  $\tilde{c}_\psi$  and  $\tilde{c}_{\psi'}$  for the functor  $\tilde{c}$  applied with respect to  $\psi$  or  $\psi'$  respectively. As we are simply shifting the moment map, it follows easily that, for any conical symplectic toric bundle over  $\psi$   $(\pi : P \rightarrow W, \omega)$ , this bundle must be a conical symplectic toric bundle over  $\psi'$  as well.

It is easy then to check that, from how  $\tilde{c}_\psi$  and  $\tilde{c}_{\psi'}$  were defined,  $\tilde{c}_\psi(\pi : P \rightarrow W, \omega)$  and  $\tilde{c}_{\psi'}(\pi : P \rightarrow W, \omega)$  are the same as symplectic stratified  $G$ -spaces; that is, they are the same



topological space with the same partition with the same symplectic form on the open dense piece. It follows there is a  $G$ -equivariant isomorphism of stratified spaces

$$f : \tilde{c}_\psi(\pi : P \rightarrow W, \omega) \rightarrow \tilde{c}_{\psi'}(\pi : P \rightarrow W, \omega)$$

restricting to a symplectomorphism on the open dense pieces and preserving the respective  $G$ -quotients to  $W$ . Of course, for moment maps  $\mu$  and  $\mu'$  for  $\tilde{c}_\psi(\pi : P \rightarrow W, \omega)$  and  $\tilde{c}_{\psi'}(\pi : P \rightarrow W, \omega)$ , we have  $\mu' \circ f = \mu + \eta$ .

Now, we begin to show that  $\tilde{c}$  is an isomorphism of presheaves. Rather than directly showing  $\tilde{c}$  is fully faithful, we show that  $\tilde{c}$  fits into a diagram of fully faithful functors. Recall that, for stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , we have the functors

$$\iota : \text{CSTB}_\psi(W) \rightarrow \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$$

(see Definition 10.6) and

$$\text{res} : \text{STSS}_\psi(W) \rightarrow \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$$

(see Definition 9.6), which both may be thought of as presheaf restrictions from  $W$  to  $W_{\text{reg}}$  followed by identifications  $\text{CSTB}_\psi(W_{\text{reg}}) = \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  and  $\text{STSS}_\psi(W_{\text{reg}}) = \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$ , respectively.

**Proposition 11.6.** For any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , the diagram:

$$\begin{array}{ccc} \text{CSTB}_\psi(W) & \xrightarrow{\iota} & \text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}}) \\ \tilde{c} \downarrow & & \downarrow c \\ \text{STSS}_\psi(W) & \xrightarrow{\text{res}} & \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}}) \end{array}$$

commutes.

*Proof.* This is more or less obvious from the details of the construction of  $\tilde{c}$ .  $\square$

**Lemma 11.7.** For any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ ,  $\text{res} : \text{STSS}_\psi(W) \rightarrow \text{STM}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$  is fully faithful.

*Proof.* Since  $\iota$  and  $c$  are both fully faithful (see Lemma 10.7), it follows  $c \circ \iota$  is as well. Hence, by Proposition 11.6,  $\text{res} \circ \tilde{c}$  is also fully faithful. It is easy to check that, since  $\text{res}$  is fully faithful (see Lemma 9.7), it must follow that  $\tilde{c}$  is fully faithful.  $\square$

Now, we show that the elements of  $\text{STSS}_\psi$  are locally isomorphic to elements of the image of  $\tilde{c}$  in a particular way.

**Lemma 11.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then for any symplectic toric stratified space  $(X, X_{\text{reg}} \sqcup \{x_\alpha\}_{\alpha \in A}, \omega, \pi : X \rightarrow W)$  over  $\psi$  and for any point  $w \in W$ , there is an open neighborhood  $U_w$  of  $w$  and a conical symplectic toric bundle  $(\varpi : P \rightarrow U_w, \eta)$  in  $\text{CSTB}_\psi(U_w)$  so that  $\tilde{c}((P, \eta))$  is isomorphic to  $(X, \omega, \pi : X \rightarrow W)|_{U_w}$ .

*Proof.* In the case where  $w \in W_{\text{reg}}$ , this is done simply by choosing a contractible open neighborhood of  $w$  small enough so that  $U_w \subset W_{\text{reg}}$ . Here,  $\text{STSS}_\psi(U_w) = \text{STM}_{\psi|_{W_{\text{reg}}}}(U_w)$  and, since  $U_w$  is contractible, all elements of  $\text{STM}_{\psi|_{W_{\text{reg}}}}(U_w)$  are isomorphic, so the image of any element of  $\text{CSTB}_\psi(U_w)$  is isomorphic to the restriction our original symplectic toric

stratified space. By Proposition 10.9,  $\text{CSTB}_\psi(U_w)$  is non-empty, so we may find such a bundle.

So we consider the case where  $w$  is a singularity of  $W$ . Then let  $U_w$  be any neighborhood of  $w$  for which there exists a singular symplectic toric cone  $(C, \omega', \nu : C \rightarrow \mathfrak{g}^*)$  with neighborhood  $V$  of the cone point  $*$  of  $C$ , and a map  $\varphi : \pi^{-1}(U_w) \rightarrow V$  satisfying the conditions described in Definition 9.1. Then for orbital moment map  $\bar{\nu} : C/G \rightarrow \mathfrak{g}^*$ ,  $\bar{\nu}|_{(C/G)_{\text{reg}}}$  is a homogeneous unimodular local embedding.

Let  $q : C \rightarrow C/G$  be a  $G$ -quotient map. As  $\varphi$  is  $G$ -equivariant, it follows that it descends to an open embedding of cornered stratified spaces  $\bar{\varphi} : U_w \rightarrow (C/G)_{\text{reg}}$  satisfying  $q \circ \varphi = \bar{\varphi} \circ \pi$  and  $\bar{\nu} \circ \bar{\varphi} + \psi(w_\alpha) = \psi$ . As discussed in Remark 11.5,  $(C, \omega', q : C \rightarrow C/G)$  is also a symplectic toric stratified space over the stratified unimodular local embedding  $\bar{\nu} + \psi(w)$ . With this in mind, it follows that  $\bar{\varphi}^*(C, \omega', q : C \rightarrow C/G)$  is a symplectic toric stratified space over  $\psi$  and, since  $\bar{\varphi}^*(C, \omega', q : C \rightarrow C/G)$  is isomorphic to  $V$ ,  $\varphi$  induces an isomorphism

$$\tilde{\varphi} : (X, \omega, \pi : X \rightarrow W)|_{U_w} \rightarrow \bar{\varphi}^*(C, \omega', q : C \rightarrow C/G)$$

in  $\text{STSS}_\psi(U_w)$ .

On the other hand, since

$$\text{hc} : \text{HSTB}_{\bar{\nu}|_{C_{\text{reg}}/G}}(C_{\text{reg}}/G) \rightarrow \text{STC}_{\bar{\nu}|_{C_{\text{reg}}/G}}(C_{\text{reg}}/G)$$

is an equivalence of categories (see Theorem 5.7), there exists a homogeneous symplectic toric bundle  $(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$  and an isomorphism of symplectic toric cones over  $\bar{\nu}|_{C_{\text{reg}}/G}$

$$f : (C_{\text{reg}}, \omega', q|_{C_{\text{reg}}} : C_{\text{reg}} \rightarrow C_{\text{reg}}/G) \rightarrow \text{hc}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$$

Following the steps defining  $\tilde{c}$ , we may consider  $(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$  as a conical symplectic toric bundle over  $\bar{\nu} : C/G \rightarrow \mathfrak{g}^*$ ; we have then that  $\tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$  is a singular symplectic toric cone with regular part the symplectic toric cone  $\text{hc}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$ . By Lemma 8.11, the isomorphism  $f$  extends to an isomorphism of symplectic toric stratified spaces over  $\bar{\nu}$

$$\tilde{f} : (C, \omega', q : C \rightarrow C/G) \rightarrow \tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta).$$

Now again as in the discussion in Remark 11.5, we may shift the moment map for  $(C, \omega', q : C \rightarrow C/G)$  and  $\tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$  by  $\psi(w)$  to obtain an isomorphism

$$\tilde{f}' : (C, \omega', q : C \rightarrow C/G) \rightarrow \tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$$

in  $\text{STSS}_{\bar{\nu} + \psi(w)}(C/G)$ ; here we also “shift”  $(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$  to a conical symplectic toric bundle over  $\bar{\nu} + \psi(w)$ .

Finally, as shown in Lemma 9.9, isomorphisms pullback as well; that is, the isomorphism  $\tilde{f}$  pulls back to an isomorphism

$$\varphi^* \tilde{f} : \varphi^*(C, \omega', q : C \rightarrow C/G) \rightarrow \varphi^* \tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta)$$

in  $\text{STSS}_\psi(U_w)$ . By Lemma 11.2,  $\varphi^* \tilde{c}(\pi : P \rightarrow C_{\text{reg}}/G, \eta) = \tilde{c}(\varphi^*(\pi : P \rightarrow C_{\text{reg}}/G, \eta))$ , so we have an isomorphism

$$(\tilde{\varphi})^{-1} \circ (\varphi^* \tilde{f})^{-1} : (X, \omega, \pi : X \rightarrow W)|_{U_w} \rightarrow \tilde{c}(\varphi^*(\pi : P \rightarrow C_{\text{reg}}/G, \eta))$$

in  $\text{STSS}_\psi(U_w)$ .

□

We need one more lemma before we can finish.

**Lemma 11.9.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then the presheaf of groupoids  $\text{STSS}_\psi : \text{Open}(W)^{op} \rightarrow \text{Groupoids}$  is a prestack (see Definition B.4).

*Proof.* Let  $U \subset W$  be any open subset and let  $(X, \omega, \pi : X \rightarrow U)$ ,  $(X', \omega', \pi' : X' \rightarrow U)$  be any two symplectic toric stratified spaces over  $\psi|_U$ . As a map of symplectic toric stratified spaces  $f : (X, \omega, \pi : X \rightarrow U) \rightarrow (X', \omega', \pi' : X' \rightarrow U)$  is, in particular, a homeomorphism  $f : X \rightarrow X'$ , it is clear that  $f$  is uniquely determined by its restrictions to any open cover of  $X$ .

So, suppose that  $\{U_\alpha\}_{\alpha \in A}$  is any open cover of  $U$ . Suppose also that we have a family of isomorphisms

$$\{f_\alpha : (X, \omega, \pi : X \rightarrow U)|_{U_\alpha} \rightarrow (X', \omega', \pi' : X' \rightarrow U)|_{U_\alpha}\}_{\alpha \in A}$$

so that, for every  $\alpha$  and  $\beta$  with  $U_\alpha \cap U_\beta$  non-empty,  $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ . Then clearly the  $f_\alpha$ 's glue together to a homeomorphism  $f : X \rightarrow X'$ . Since  $f_\alpha|_{X_{\text{reg}} \cap U_\alpha}$  is a symplectomorphism for each  $\alpha$ , it follows that  $f|_{X_{\text{reg}}}$  must be a symplectomorphism as well. Finally, as each  $f_\alpha$  preserves the quotients  $\pi|_{U_\alpha}$  and  $\pi'|_{U_\alpha}$ , the glued together map  $f$  must preserve these quotients as well. Therefore, this  $f : X \rightarrow X'$  in fact defines an isomorphism  $f : (X, \omega, \pi) \rightarrow (X', \omega', \pi')$ .

Thus, the presheaf of sets  $\underline{\text{Hom}}(X, X')$  defined by  $U \mapsto \text{Hom}(X|_U, X'|_U)$  (again, as in Definition B.4) is a sheaf.  $\square$

We may now prove the following theorem.

**Theorem 11.10.** For any stratified unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ , the map of presheaves  $\tilde{c} : \text{CSTB}_\psi \rightarrow \text{STSS}_\psi$  is an isomorphism of presheaves.

*Proof.* We've shown in Lemma 11.7 that  $\tilde{c}_U : \text{CSTB}_\psi(U) \rightarrow \text{STSS}_\psi(U)$  is fully faithful for every  $U$  and in Lemma 11.9 that  $\text{STSS}_\psi$  is a prestack. In Proposition B.9, we showed that  $\text{CSTB}_\psi$  is a stack. Therefore, with Lemma 11.8, we have that  $\tilde{c} : \text{CSTB}_\psi \rightarrow \text{STSS}_\psi$  satisfies the hypotheses of Lemma B.11 and is an isomorphism of presheaves.  $\square$

## 12. CHARACTERISTIC CLASSES FOR SYMPLECTIC TORIC STRATIFIED SPACES WITH ISOLATED SINGULARITIES

Now we describe a set of characteristic classes that will help identify the isomorphism classes of  $\text{CSTB}_\psi(W)$  and hence, via our equivalence of categories  $\tilde{c} : \text{CSTB}_\psi(W) \rightarrow \text{STSS}_\psi(W)$ , the isomorphism classes of  $\text{STSS}_\psi(W)$ . This is done by classifying conical symplectic toric bundles over a unimodular local embedding  $\psi : W \rightarrow \mathfrak{g}^*$ . We use the same general method as in [13], distinguishing elements of  $\text{CSTB}_\psi(W)$  by their isomorphism of principal  $G$ -bundle together with a piece of “horizontal data”.

The “horizontal data” for a conical symplectic toric bundle will take the following form.

**Definition 12.1.** Let  $(W, W_{\text{reg}} \sqcup_{\alpha \in A} \{w_\alpha\})$  be a cornered stratified space. Then say a form  $\beta \in \Omega^2(W_{\text{reg}}, \mathbb{R})$  is a **good form on  $W$**  (for the purposes of this paper) if there exist a neighborhood  $U$  of  $w_\alpha$  for each  $\alpha \in A$  such that  $\beta|_{U_{\text{reg}}}$  is exact. Clearly, any form cohomologous to  $\beta$  must also satisfy this condition, so it makes sense to denote by  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$  the set of all classes of good forms. This is, of course, a subspace.

**Notation 12.2.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding and let  $\pi : P \rightarrow W_{\text{reg}}$  be a principal  $G$ -bundle. Then we will denote by  $[(P, \cdot)]$  the subset of isomorphism classes

$$\{[(\pi : P \rightarrow W_{\text{reg}}, \omega)] \in \pi_0 \text{CSTB}_\psi(W)\}$$

of  $\pi_0 \text{CSTB}_\psi(W)$ . In other words, these are the classes in  $\pi_0 \text{CSTB}_\psi(W)$  containing conical symplectic toric bundles with principal bundle  $\pi : P \rightarrow W_{\text{reg}}$  and any valid  $G$ -invariant symplectic form  $\omega$  on  $P$ .

**Lemma 12.3.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding and let  $\pi : P \rightarrow W_{\text{reg}}$  be a principal  $G$ -bundle and define  $\mu := \psi \circ \pi$ . Then there is a bijection  $\mathbf{c}_{\text{hor}} : [(P, \cdot)] \rightarrow \mathcal{C}$ . This is induced by, but independent of, a choice of exact 2-form  $\eta$  for which  $(P, \eta) \in \text{CSTB}_\psi(W)$ .

*Proof.* By Proposition 10.9, we have that there exists a  $G$ -invariant symplectic form  $\eta$  on  $P$  with respect to which  $(P, \eta)$  is a conical symplectic toric bundle over  $\psi$ . By Corollary 2.13 we have that, for any other  $G$ -invariant symplectic form  $\omega$  on  $P$  with moment map  $\mu$ ,  $\omega - \eta$  is a basic closed form. Then let  $f$  be the function

$$G\text{-invariant symplectic forms on } P \text{ with moment map } \mu \xrightarrow{f} \text{Closed 2-forms on } W_{\text{reg}} \quad (12.1)$$

with  $f(\omega)$  the 2-form satisfying  $\eta - \omega = \pi^* f(\omega)$ . This is well-defined as  $\pi$  is a submersion.

Suppose  $\omega$  is a  $G$ -invariant symplectic form on  $P$  with moment map  $\mu$  so that  $(P, \omega) \in \text{CSTB}_\psi(W)$ . Then there exist neighborhoods  $U_\alpha$  of each  $w_\alpha$  for which  $(P, \omega)|_{U_\alpha}$  is symplectomorphic to a neighborhood of  $-\infty$  in some symplectic toric cone. In particular, this means  $\omega|_{P|_{U_{\alpha\text{reg}}}}$  must be exact. Therefore,  $\omega - \eta$  must be exact over (a possibly smaller) deleted neighborhood of each singularity. Thus, the cohomology class  $[f(\omega)]$  is actually an element of  $\mathcal{C}$ .

Now, for  $\omega$  and  $\omega'$   $G$ -invariant symplectic forms on  $P$  for which  $(P, \omega), (P, \omega') \in \text{CSTB}_\psi(W)$ , if  $\omega - \omega'$  is exact, then Lemma 2.14 tells us that  $\iota(P, \omega)$  and  $\iota(P, \omega')$  are isomorphic in  $\text{STB}_{\psi|_{W_{\text{reg}}}}(W_{\text{reg}})$ . Recall that  $\iota$  (see Definition 10.6) is fully faithful (see Lemma 10.7); therefore, if  $\iota(P, \omega)$  and  $\iota(P, \omega')$  are isomorphic, then  $(P, \omega)$  and  $(P, \omega')$  must be isomorphic as well.

It follows from the previous two paragraphs that  $f$  descends to an injective function:

$$\mathbf{c}_{\text{hor}} : [(P, \cdot)] \rightarrow \mathcal{C} \quad \mathbf{c}_{\text{hor}}([P, \omega]) := [f(\omega)].$$

Finally, suppose  $[\beta]$  is any class in  $\mathcal{C}$  and fix a good form  $\beta \in \Omega^2(W_{\text{reg}}, \mathbb{R})$  to represent this class. Then, by definition, for each singularity  $w_\alpha$  of  $W$ , there is an open subset  $U_\alpha$  of  $w_\alpha$  and a form  $\gamma_\alpha$  on  $U_{\alpha\text{reg}}$  with  $\beta|_{U_{\alpha\text{reg}}} = d\gamma_\alpha$ . Let  $U_0 \subset W_{\text{reg}}$  be an open subset so that:

- $\{U_0, \{U_\alpha\}_{\alpha \in A}\}$  is an open cover of  $W$
- For each  $\alpha$ , there is an open neighborhood  $V_\alpha \subset U_\alpha$  of  $w_\alpha$  with  $V_\alpha \cap U_0 = \emptyset$

Let  $\{\rho_0, \{\rho_\alpha\}_{\alpha \in A}\}$  be a partition of unity for this open cover. Define  $\gamma := \sum_\alpha \rho_\alpha \gamma_\alpha$ . Then, for  $\beta' := \beta - d\gamma$ , we have  $[\beta'] = [\beta]$ . Since  $\beta'|_{V_{\alpha\text{reg}}} = 0$  for every  $\alpha$ , it follows that  $(P, \eta + \pi^* \beta')$  is an element of  $\text{CSTB}_\psi(W)$ . Therefore, we have that  $\mathbf{c}_{\text{hor}}(P, \eta + \pi^* \beta') = [\beta]$  and so  $\mathbf{c}_{\text{hor}}$  must be surjective.

Now, suppose  $\eta'$  is another choice of exact 2-form and write  $f'$  for the function defined with respect to  $\eta'$  analogous to the function  $f$  in Equation (12.1) above. Let  $\omega$  be a 2-form for which  $(P, \omega) \in \text{CSTB}_\psi(W)$ . Then, by definition, we have  $\eta + \pi^* f(\omega) = \eta' + \pi^* f'(\omega)$ .

But then  $\eta - \eta' = \pi^*(f'(\omega) - f(\omega))$ . Since  $\eta$  and  $\eta'$  are both exact, we may conclude that  $[f'(\omega)] = [f(\omega)]$ . Therefore, the definition for  $\mathbf{c}_{\text{hor}}$  is independent of the choice of closed 2-form  $\eta$  used to define it.  $\square$

We may now classify conical symplectic toric bundles over any stratified unimodular local embedding.

**Proposition 12.4.** Let  $G$  be a torus with Lie algebra  $\mathfrak{g}$  and let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then there is an isomorphism of presheaves

$$(\mathbf{c}_1, \mathbf{c}_{\text{hor}}) : \pi_0 \mathbf{CSTB}_\psi \rightarrow H^2(\cdot, \mathbb{Z}_G) \times \mathcal{C}$$

where  $\mathbb{Z}_G := \ker(\exp : \mathfrak{g} \rightarrow G)$  and  $H^2(\cdot, \mathbb{Z}_G)$  is the presheaf of sets

$$H^2(\cdot, \mathbb{Z}_G) : \mathbf{Open}(W)^{\text{op}} \rightarrow \mathbf{Sets} \quad U \mapsto H^2(U_{\text{reg}}, \mathbb{Z}_G)$$

*Proof.* We have essentially the same ingredients as the similar Proposition 5.1 of [13] had for symplectic toric bundles, so we approach the problem in the same way.

Fix  $U$  any open subset of  $W$ . For  $(P, \omega) \in \mathbf{CSTB}_\psi(U)$ , define

$$(\mathbf{c}_1, \mathbf{c}_{\text{hor}})([P, \omega]) := (\mathbf{c}_1(P), \mathbf{c}_{\text{hor}}(\omega))$$

for  $\mathbf{c}_{\text{hor}}$  the bijection  $\mathbf{c}_{\text{hor}} : [(P, \cdot)] \rightarrow \mathcal{C}$  of Lemma 12.3. To see that this is well-defined, suppose  $\varphi : (\pi' : P' \rightarrow U_{\text{reg}}, \omega') \rightarrow (\pi : P \rightarrow U_{\text{reg}}, \omega)$  is an isomorphism in  $\mathbf{CSTB}_\psi(U)$ . Pick an exact 2-form  $\eta$  on  $P$  defining function  $f$  as in equation (12.1) for which  $\eta - \omega = \pi^*f(\omega)$  and  $\mathbf{c}_{\text{hor}}([\omega]) = [f(\omega)]$ .

Then  $\varphi^*\eta$  is an exact symplectic form on  $P'$  with moment map  $\psi \circ \pi'$ . For  $f'$  the function with  $\varphi^*\eta - \omega' = \pi'^*f'(\omega')$ , we have

$$\varphi^*(\pi^*(f'(\omega'))) = \pi'^*f'(\omega') = \varphi^*\eta - \omega' = \varphi^*(\eta - \omega)$$

Since  $\varphi$  is a diffeomorphism, it follows  $\pi^*f'(\omega') = \eta - \omega = \pi^*f(\omega)$ . Therefore,  $f'(\omega') = f(\omega)$ , so  $\mathbf{c}_{\text{hor}}(P', \omega') = \mathbf{c}_{\text{hor}}(P, \omega)$ . Thus,  $(\mathbf{c}_1, \mathbf{c}_{\text{hor}})([P, \omega])$  is well-defined.

As noted in Remark 6.7,  $\mathbf{c}_1$  commutes with restrictions. Suppose  $V$  is an open subset of  $U$  and let  $(P, \omega)$ ,  $\eta$  and  $f$  be as above. Then clearly  $\eta|_{P|_{V_{\text{reg}}}}$  is an exact 2-form from which we may define  $\mathbf{c}_{\text{hor}}((P, \omega)|_V)$ . So since  $\eta - \omega = \pi^*f(\omega)$ , we have

$$\eta|_{P|_{V_{\text{reg}}}} - \omega|_{P|_{V_{\text{reg}}}} = (\eta - \omega)|_{P|_{V_{\text{reg}}}} = (\pi^*f(\omega))|_{P|_{V_{\text{reg}}}} = \pi^*(f(\omega)|_{V_{\text{reg}}})$$

It follows that  $\mathbf{c}_{\text{hor}}((P, \omega)|_V) = (\mathbf{c}_{\text{hor}}(P, \omega))|_V$ . Therefore,

$$(\mathbf{c}_1, \mathbf{c}_{\text{hor}})((P, \omega)|_V) = (\mathbf{c}_1(P|_{V_{\text{reg}}}), \mathbf{c}_{\text{hor}}(\omega|_{P|_{V_{\text{reg}}}})) = (\mathbf{c}_1(P)|_V, \mathbf{c}_{\text{hor}}(\omega)|_V) = ((\mathbf{c}_1, \mathbf{c}_{\text{hor}})(P, \omega))|_V$$

and so  $(\mathbf{c}_1, \mathbf{c}_{\text{hor}})$  is a map of presheaves.

Now, suppose  $(\pi : P \rightarrow U_{\text{reg}}, \omega)$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')$  are two elements of  $\mathbf{CSTB}_\psi(U)$  so that  $(\mathbf{c}_1, \mathbf{c}_{\text{hor}})(P, \omega) = (\mathbf{c}_1, \mathbf{c}_{\text{hor}})(P', \omega')$ . In particular, this means there is an isomorphism of principal bundles  $\varphi : P' \rightarrow P$  since  $\mathbf{c}_1(P) = \mathbf{c}_1(P')$ . Let  $\eta$  be an exact 2-form on  $P$  for which  $\eta - \omega = \pi^*\mathbf{c}_{\text{hor}}(P, \omega)$ . Then, as above,  $\varphi^*\eta$  is an exact symplectic form on  $P'$  with moment map  $\psi \circ \pi'$ , so:

$$\varphi^*\eta - \omega' = \pi'^*\mathbf{c}_{\text{hor}}(P', \omega') = \varphi^*(\pi^*(\mathbf{c}_{\text{hor}}(P, \omega))) = \varphi^*(\pi^*(\mathbf{c}_{\text{hor}}(P, \omega) + \pi^*(d\gamma)))$$

For some 1-form  $\gamma$  on  $U_{\text{reg}}$ . Then

$$\varphi^*\eta - \omega' = \varphi^*(\eta - \omega) + \varphi^*(d\pi^*\gamma) = \varphi^*\eta - \varphi^*\omega + d(\pi'^*\gamma)$$

Thus  $\varphi^*\omega - \omega' = d(\pi'^*\gamma)$ . From Lemma 2.14, it follows there is a gauge transformation  $\phi : P \rightarrow P$  with  $\phi^*(\omega' + d(\pi'^*\gamma)) = \omega'$ . Thus,  $\phi \circ \varphi : P' \rightarrow P$  is an isomorphism between  $(P, \omega)$  and  $(P', \omega')$  in  $\mathbf{CSTB}_\psi(U)$ . Therefore,  $(c_1, c_{\text{hor}})$  is injective over each open  $U$ .

Finally, suppose  $(c, [\beta]) \in H^2(U; \mathbb{Z}_G) \times \mathcal{C}$ . Then there exists principal bundle  $\pi : P \rightarrow U_{\text{reg}}$  with  $c_1(P) = c$ . Again, by Proposition 10.9, there exists  $\eta$  an exact symplectic form on  $P$  with moment map  $\psi \circ \pi$ . Then, as shown in Lemma 12.3, there is a form  $\beta'$  with  $[\beta] = [\beta']$  for which  $(P, \eta + \pi^*\beta') \in \mathbf{CSTB}_\psi(U)$ . By design,  $(c_1, c_{\text{hor}})(P, \eta + \pi^*\beta') = (c, [\beta])$  and therefore  $(c_1, c_{\text{hor}})$  is surjective over each open  $U$ .  $\square$

Now, we may easily prove the following classification.

**Theorem 1.1.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then there is a subspace  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$  so that the isomorphism classes of symplectic toric stratified spaces with isolated singularities  $(X, \omega, \mu : X \rightarrow \mathfrak{g}^*)$  with  $G$ -quotient map  $\pi : X \rightarrow W$  and orbital moment map  $\psi$  are in natural bijective correspondence with the cohomology classes

$$H^2(W_{\text{reg}}, \mathbb{Z}_G) \times \mathcal{C}$$

where  $\mathbb{Z}_G$  is the integral lattice of  $G$ , the kernel of the map  $\exp : \mathfrak{g} \rightarrow G$ .

*Proof.* This bijection is given by the composite of

$$(\pi_0 \tilde{\mathbf{c}})^{-1} : \pi_0 \mathbf{STSS}_\psi(W) \rightarrow \pi_0 \mathbf{CSTB}_\psi(W)$$

(which is a bijection since  $\tilde{\mathbf{c}}$  is an equivalence of categories, by Theorem 11.10) and

$$(c_1, c_{\text{hor}}) : \pi_0 \mathbf{CSTB}_\psi(W) \rightarrow H^2(W_{\text{reg}}, \mathbb{Z}_G) \times \mathcal{C}$$

(a bijection by Proposition 12.4).  $\square$

Now we've identified characteristic classes for symplectic toric stratified spaces with isolated singularities, it would be good to understand how to actually calculate what the subspace  $\mathcal{C}$  actually looks like for a general cornered stratified space. For this, we turn to relative de Rham cohomology, as presented by Bott and Tu [3]. A full description is given in Appendix C. It turns out we can identify a subset  $\overline{W}$  of  $W_{\text{reg}}$  so that the image of  $H^2(W_{\text{reg}}, \overline{W})$  into  $H^2(W_{\text{reg}}, \mathbb{R})$  via the long exact sequence of relative de Rham cohomology (see Equation (C.1)) is exactly  $\mathcal{C}$ .

We identify this subset  $\overline{W}$  with the following lemma.

**Lemma 12.5.** For any cornered stratified space  $(W, W_{\text{reg}} \sqcup_{\alpha \in I} \{w_\alpha\})$ , there exists a subset  $\overline{W}$  of  $W_{\text{reg}}$  so that, if  $\beta$  is any good form on  $W$  (see Definition 12.1), then  $\beta$  is exact on all of  $\overline{W}$ . More precisely, we can identify a subset  $\overline{W}$  of  $W_{\text{reg}}$  so that forms exact in (deleted) neighborhoods of singularities of  $W$  must be exact on  $\overline{W}$ .

*Proof.* Fix a singularity  $w_0$ . Then a neighborhood  $U$  of  $w_0$  in  $W$  is isomorphic to  $c(L)$  for some compact manifold with corners  $L$ , meaning  $U' := U \setminus \{w_0\}$  is a deleted neighborhood of  $w_0$  in  $W_{\text{reg}}$  diffeomorphic to  $L \times \mathbb{R}$ . Now, suppose  $\beta$  is any good form on  $W$ . Then there exists  $V \subset U'$  on which  $\beta$  is exact and so that  $V \sqcup \{w_0\}$  is a neighborhood of  $w_0$  in  $W$ . Thus, as  $L$  is compact, there exists  $\epsilon$  so that  $L \times (-\infty, \epsilon) \subset V$ .

Pick  $\tau < \epsilon$ . Then the map  $f_\tau : L \rightarrow L \times \mathbb{R}$  defined by  $f_\tau(l) := (l, \tau)$  is part of a homotopy equivalence with the projection  $L \times \mathbb{R} \rightarrow L$ . As  $\beta$  is exact on  $L \times \{\tau\}$ , it follows that  $\beta$  is

exact on *all* of  $L \times \mathbb{R}$ . Thus, any good  $\beta$  must be exact on  $U'$ . So we may assemble  $\overline{W}$  as the union of all (deleted) conical neighborhoods of the singularities of  $W$ .  $\square$

So, in light of the previous lemma, we may identify  $\mathcal{C}$  as those elements of  $H^2(W, \mathbb{R})$  that are exact on  $\overline{W}$ . Now that we have one standard neighborhood on which all good forms are exact, we may proceed more easily.

**Proposition 12.6.** Let  $W$  be any cornered stratified space and let  $f : \overline{W} \rightarrow W_{\text{reg}}$  the inclusion of the open subspace  $\overline{W}$  of  $W_{\text{reg}}$  identified above. Then, for  $H^2(W_{\text{reg}}, \overline{W})$  the relative de Rham cohomology group (see Appendix C),  $\mathcal{C} \subset H^2(W_{\text{reg}}, \mathbb{R})$  is exactly the image of  $H^2(W_{\text{reg}}, \overline{W})$  under the map  $\tilde{\pi}([\alpha, \beta]) := [(\beta)]$  of the long exact sequence

$$\dots \xrightarrow{f^*} H^1(\overline{W}) \xrightarrow{\tilde{i}} H^2(W_{\text{reg}}, \overline{W}) \xrightarrow{\tilde{\pi}} H^2(W_{\text{reg}}) \xrightarrow{f^*} H^2(\overline{W}) \longrightarrow \dots$$

(see Proposition C.2).

*Proof.* If  $[\beta]$  is a class of good forms in  $\mathcal{C}$ , then Lemma 12.5 tells us that  $\beta$  is exact on  $\overline{W}$ . So there exists  $\gamma \in \Omega^1(\overline{W})$  such that  $d\gamma = \beta|_{\overline{W}}$ . Thus  $[(\beta, \gamma)] \in H^2(W_{\text{reg}}, \overline{W})$  and  $\tilde{\pi}([( \beta, \gamma)]) = [\beta]$ .

Conversely, any class of good forms  $[\beta]$  is exact on  $\overline{W}$  and so there is a 1-form  $\gamma$  on  $\overline{W}$  with  $\beta|_{\overline{W}} = d\gamma$  and so  $[(\beta, \gamma)] \in H^2(W_{\text{reg}}, \overline{W})$  and  $\tilde{\pi}([\beta, \gamma]) = [\beta]$ .  $\square$

### 13. RELATION TO EARLIER WORK AND EXAMPLES

In this section, we demonstrate how the main theorems of this paper are extensions of previous work. We also provide some examples to which we may apply our main theorems.

First, we recall the work of Lerman in [15] classifying contact toric manifolds. His definition of contact toric manifolds matches ours: co-oriented contact manifolds  $(B, \xi)$  with an effective contact action of a torus  $G$  satisfying  $2 \dim(G) = \dim(B) + 1$  with isomorphisms  $G$ -equivariant co-orientation preserving contactomorphisms.

To discuss moment maps, Lerman defines two notions: the notion of a *moment cone* and the notion of a *good cone*. For a contact toric manifold  $(B, \xi)$  and for any  $G$ -invariant contact form  $\alpha$  with  $\alpha$  moment map  $\mu_\alpha : B \rightarrow \mathfrak{g}^*$ , the moment cone for  $(B, \xi)$  is the cone:

$$C = \{0\} \cup \{e^t \mu_\alpha(b) \mid b \in B, t \in \mathbb{R}\}$$

A good cone is a cone of  $\mathfrak{g}^*$  that may be written as the intersection

$$\bigcap_{i=1}^{\dim(G)} \{\eta \in \mathfrak{g}^* \mid \langle \eta, v_i \rangle \geq 0\}$$

for  $\{v_i\}$  a minimal subset of primitive vectors in  $\mathbb{Z}_G$  so that, for any subset  $\{v_{i_j}\}_{1 \leq j \leq k}$  with  $0 < k < \dim(G)$  for which

$$C \cap \bigcap_{j=1}^k \{\eta \in \mathfrak{g}^* \mid \langle \eta, v_{i_j} \rangle = 0\}$$

is a face of  $C$ , the subset  $\{v_{i_j}\}_{1 \leq j \leq k}$  is the integral basis of a sublattice of  $\mathbb{Z}_G$  corresponding to a subtorus of  $G$ .

He then proves the following theorem.

**Theorem 13.1** (Theorem 2.18, [15]). Compact connected contact toric manifolds are classified as follows.

- (1) In dimension 3, contact toric manifolds with free torus actions are, up to isomorphism, exactly the product  $S^1 \times \mathbb{T}^2$  with a contact form  $\alpha = \cos(nt)d\theta_1 + \sin(nt)d\theta_2$  for  $n$  a positive integer (here,  $(t, \theta_1, \theta_2)$  are the coordinates for  $S^1 \times \mathbb{T}^2$ ).
- (2) Compact contact toric manifolds of dimension 3 with a torus action that isn't free are diffeomorphic to lens spaces and are classified by pairs of rational numbers  $r, q$  with  $0 \leq r < 1$ ,  $r < q$ .
- (3) Compact contact toric manifolds of dimension greater than 3 are always principal  $G$ -bundles over the sphere  $S^d$  for  $d = \dim G - 1$  with co-oriented contact structure uniquely determined by this principal bundle structure.
- (4) The moment cone of a compact contact toric manifold of dimension greater than 3 with a non-free torus action is a good cone and, for any good cone  $C$ , there is a unique isomorphism class of compact contact toric manifolds with moment cone  $C$ .

In dimension 3, Lerman's result is a reflection of the fact that orbital moment maps for 4 dimensional connected symplectic toric cones with compact links are, in the free action case,  $n$ -fold coverings of  $\mathbb{R}^2 \setminus \{0\}$  and, in the non-free action case, restrictions of such coverings. In either case, the orbit spaces are diffeomorphic to either  $\mathbb{R} \times S^1$  or  $\mathbb{R} \times [0, 1]$  and therefore have no 2<sup>nd</sup> degree cohomology. It follows by Theorem 1.2 that all symplectic toric cones (and hence all contact toric manifolds) with the same orbital moment map are isomorphic, which, modulo the actual description of these classes, is the content of points (1) and (2) of the above theorem.

In the case where the dimension of the contact toric manifold in question is greater than 3, the orbital moment map of the corresponding symplectic toric cone is injective (this follows from Theorem 4.2 of [15] which gives that the fibers of the moment map must be connected). So, in the case where the torus action is free, the corresponding symplectic toric cone is, in fact a principal  $G$ -bundle with a free  $\mathbb{R}$  action and, as shown in Proposition 4.10, there is a unique symplectic structure for each such topological structure. It follows that, by quotienting by against the  $\mathbb{R}$  action, we obtain a principal  $G$ -bundle over  $S^{\dim(G)-1}$  with a unique contact structure.

In the case where the torus action is not free, we rely on the fact that the moment cone in this case is, in particular, homeomorphic to the cone on a convex subset of  $S(\mathfrak{g}^*)$  (see Theorem 1.2 of [14]). Thus, the moment cone, which as we mentioned before is diffeomorphic to the orbit space, is contractible and therefore uniquely determines the symplectic toric cone and therefore contact toric manifold isomorphism class.

In the case of symplectic toric stratified spaces, the precedent in the compact case comes from Burns, Guillemin, and Lerman [5]. Their definition agrees with ours with the exceptions that

- (1) The open dense piece of a symplectic toric stratified space with isolated singularities is allowed to be a symplectic toric orbifold; and
- (2) The symplectic toric cones on which neighborhoods of singularities are modelled must be Reeb type: for  $(M, \omega)$  such a symplectic toric cone with homogeneous moment map  $\mu$  modeling an open neighborhood of singular point  $x_\alpha$ , we assume there exists vector  $Y \in \mathfrak{g}$  such that  $\langle \mu - \mu(x_\alpha), Y \rangle > 0$ .



While we do not assume the former condition for our symplectic toric stratified spaces with isolated singularities, we are able to drop the latter condition of Reeb type. So, assume we are working in the case where the strata of a compact symplectic toric stratified space are manifolds.

Then Burns, Guillemin, and Lerman proved the following (which we abridge and adjust to the language of this paper)

**Theorem 13.2** (Theorem 1, [5]). For  $(X, \omega, \mu)$  a compact connected symplectic toric stratified space with isolated singularities:

- (1)  $\mu(X)$  is a rational polytope, simple except possibly at its vertices;
- (2) The fibers of  $\mu$  are  $G$ -orbits; and
- (3)  $(X, \omega, \mu)$  is uniquely determined up to isomorphism by  $\mu(X)$ .

Here, a rational polytope is simple if its facets (i.e., its codimension 1 faces) lie in general position. In other words, we require that the non-empty intersections of any  $k$  facets yield codimension  $k$  faces of the polytope. Implicitly, this forbids non-empty intersections of more than  $\dim(\mathfrak{g}^*)$  facets. As an example, note that, for  $\mathfrak{g}^* \cong \mathbb{R}^3$ , the cube is a simple rational polytope while the octahedron is simple only away from the vertices.

We will not bother to repeat the proofs of items 1 and 2 of the above theorem. Instead, we focus on item 3. First, note that if  $\Delta$  is a rational polytope that is simple everywhere except perhaps at the vertices, then the inclusion  $\iota : \Delta \rightarrow \mathfrak{g}^*$  is a stratified unimodular local embedding. Then since  $\Delta$  is convex, it follows that  $H^2(W_{\text{reg}}, \mathbb{Z}_G) = H^2(W_{\text{reg}}, \mathbb{R}) = 0$ . Therefore, by Theorem 11.10, we have that all elements of  $\text{STSS}_\iota(\Delta)$  are isomorphic. Thus, compact connected symplectic toric stratified spaces are uniquely determined by the image of their moment map.

In the remainder of this section, we provide some examples illustrating some stratified unimodular local embeddings as well as some applications of our classification theorems.

**Example 13.3.** Let  $G = \mathbb{T}^2$  and let  $\psi : \mathbb{C} \rightarrow \mathfrak{g}^*$  be the exponential map (i.e., taking  $z \in \mathbb{C}$  to  $e^z$  in  $\mathbb{C} \cong \mathfrak{g}^*$ ). Then  $\psi$  is a homogeneous unimodular local embedding and there is exactly one isomorphism class of symplectic toric cones over  $\psi$ . This class is represented by the symplectization of the contact structure  $(\mathbb{R} \times \mathbb{T}^2, \xi)$  for  $\xi$  the contact structure the kernel of the form  $\alpha := \cos(2\pi t)d\theta_1 + \sin(2\pi t)d\theta_2$ , where  $(t, \theta_1, \theta_2)$  represent coordinates for  $\mathbb{R} \times \mathbb{T}^2$ .

**Example 13.4.** Unlike symplectic toric manifolds, symplectic toric stratified spaces may be compact and connected but have non-convex images. Indeed, let  $\Delta$  be the convex hull of  $(1, 1)$ ,  $(-1, 1)$ ,  $(1, -1)$ , and  $(-1, -1)$  and let  $\Delta'$  be  $\Delta$  without the fourth quadrant; i.e.,

$$\Delta' = \Delta \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq 0 \text{ or } y \geq 0\}$$

For  $\Delta'$  the stratified space with strata 0 and  $\Delta' \setminus \{0\}$ , it follows that the inclusion  $\psi : \Delta' \rightarrow \mathbb{R}^2$  is a stratified unimodular local embedding. Since  $\Delta'$  is contractible, it follows that there is a unique class of symplectic toric stratified space with this orbital moment map.

**Example 13.5.** Let  $G = \mathbb{T}^3$  and, for  $\{v_1, v_2, v_3\}$  a basis of the integral lattice of  $\mathfrak{g}^*$ , let  $\Delta$  be the octahedron in  $\mathfrak{g}^*$  that is the convex hull of  $\{\pm v_1, \pm v_2, \pm v_3\}$ . Let  $\iota : \Delta \rightarrow \mathfrak{g}^*$  be the inclusion of  $\Delta$  into  $\mathfrak{g}^*$ . Then clearly  $\iota$  is a stratified unimodular local embedding, where we think of  $\Delta$  as a cornered stratified space with its vertices as singularities. As  $\Delta$  is still convex when we remove its vertices, we have that  $H^2(\Delta_{\text{reg}}, \mathbb{R}) = H^2(\Delta_{\text{reg}}, \mathbb{Z}_G) = 0$  and so there is a unique symplectic toric stratified space over  $\iota$ .

However, if we look instead at  $\Delta_0 := \Delta \setminus \{0\}$  with embedding  $j : \Delta_0 \rightarrow \mathfrak{g}^*$ , we have that  $H^2(\Delta_{0\text{reg}}, \mathbb{Z}_G) \cong H^2(S^2, \mathbb{Z}_G) \neq 0$ . Additionally, note that each connected component of  $\overline{\Delta_0}$  chosen as in Lemma 12.5 is contractible; hence,  $H^2(\overline{\Delta_0}, \mathbb{R}) = 0$ , meaning  $\mathcal{C} = H^2(\Delta_{0\text{reg}}, \mathbb{R}) \cong H^2(S^2, \mathbb{R})$ . Thus, the elements of  $\text{STSS}_j(\Delta_0)$  are classified by the cohomology classes  $H^2(\Delta_{0\text{reg}}, \mathbb{Z}_G) \times H^2(S^2, \mathbb{R}) \neq 0$ . So while there is only one symplectic toric stratified space with moment map image  $\Delta$  (up to isomorphism), there are many isomorphism classes of symplectic toric stratified spaces with moment map image  $\Delta_0$ .

## Appendices

For this paper, we provide three appendices. Appendix A describes the well-known relationship between symplectic toric cones and contact toric manifolds. Appendix B gives a description of stacks (or, more aptly for this paper, strict sheaves of groupoids). Of note is Lemma B.11 which demonstrates how one need only show some local isomorphism conditions certain maps of presheaves to conclude that this map is in fact an isomorphism. Finally, Appendix C describes relative de Rham cohomology as well as the associated long exact sequence associated to a pair.

### APPENDIX A. SYMPLECTIC CONES AND CONTACT MANIFOLDS

This appendix gives the definition of and details about symplectic cones and contact toric manifolds used throughout the paper. Sources for this information include [15] and [16].

While we found it most convenient to use Definition 3.1 as our definition for what it meant to be a symplectic cone, an alternative characterization, sometimes also used as a definition, is often quite useful.

**Proposition A.1.** Let  $(M, \omega)$  be a symplectic cone and let  $\Xi$  be the vector field generating the action of  $\mathbb{R}$  on  $M$ . Then  $L_\Xi \omega = \omega$ .

Alternatively, if there exists vector field  $\Xi$  on  $(M, \omega)$  generating a free and proper action of  $\mathbb{R}$  and satisfying  $L_\Xi \omega = \omega$ , then  $(M, \omega)$  is a symplectic cone.

This vector field is known as the *Liouville* or *expanding* vector field.

Symplectic cones may be naturally associated with (co-oriented) contact structures on their base. For completeness, we recall the definition of these structures.

**Definition A.2.** Let  $B$  be a manifold. Then a **contact form** is a 1-form  $\alpha$  on  $B$  such that  $\xi = \ker(\alpha)$  is a 1-dimensional distribution on  $B$  with the property that  $(\xi, d\alpha|_\xi)$  is a symplectic vector bundle over  $B$ . Say that two contact forms  $\alpha$  and  $\alpha'$  are in the same **conformal class** of contact forms if there is a function  $f \in C^\infty(B)$  such that  $e^f \alpha = \alpha'$ .

Call a pair  $(B, \xi)$  of manifold with codimension 1 distribution  $\xi$  a **co-orientable contact manifold** if there exists a contact form  $\alpha$  with  $\ker(\alpha) = \xi$ . Call a co-orientable contact manifold  $(B, \xi)$  together with a choice of conformal class a **co-oriented contact manifold**.

A map of co-oriented manifolds  $\varphi : (B, \xi) \rightarrow (B', \xi')$  is a smooth map  $\varphi : B \rightarrow B'$  so that, for  $\alpha$  and  $\alpha'$  representatives of the conformal class for  $(B, \xi)$  and  $(B', \xi')$  respectively,  $\varphi^* \alpha'$  and  $\alpha$  are in the same conformal class. A **contactomorphism** between co-oriented  $(B, \xi)$  and  $(B', \xi')$  is a diffeomorphism  $\varphi : B \rightarrow B'$  so that  $\varphi$  and  $\varphi^{-1}$  are both maps of co-oriented contact manifolds.

**Remark A.3.** The reasoning behind the name co-orientable is the fact that, for  $(B, \xi)$  a co-orientable contact manifold, the line bundle  $\xi^o$ , the annihilator of  $\xi$  in  $T^*B$ , is orientable. Thinking of the 1-form  $\alpha$  as a section  $\alpha : B \rightarrow T^*B$ , a contact 1-form for  $\xi$  functions as a nowhere zero section trivializing  $\xi^o$ . It follows that  $\xi^o \setminus 0$  ( $\xi^o$  without its zero section) has two components.

In the case where  $(B, \xi)$  is co-oriented, we label by  $\xi_+^o$ , called *the symplectization of  $(B, \xi)$* , the component of  $\xi^o \setminus 0$  selected by the conformal class (co-)orienting  $(B, \xi)$ . Then, for any  $b \in B$  and  $\eta \in T_b^*B$ , the action  $t \cdot \eta := e^t \eta$  is free and proper with quotient map the restriction of the natural projection  $T^*B \rightarrow B$  to  $\xi_+^o$ .

One can show that we may restrict the canonical symplectic form on  $T^*B$  to a symplectic form on  $\xi_+^o$  and it is easy to see then that, with respect to this form and the previously mentioned  $\mathbb{R}$  action,  $\xi_+^o$  is a symplectic cone.

The relationship between a co-oriented contact manifold and its symplectization can be generalized to any symplectic cone.

**Proposition A.4.** Let  $(\pi : M \rightarrow B, \omega)$  be a symplectic cone (for  $\pi : M \rightarrow B$  the  $\mathbb{R}$ -quotient). Then

- (1)  $B$  has a natural, co-oriented contact structure  $\xi$ ;
- (2) Given a trivialization of  $M$  as a principal  $\mathbb{R}$ -bundle over  $B$   $\varphi : M \rightarrow B \times \mathbb{R}$ ,  $\varphi^* \omega = d(e^t \alpha)$  for contact form  $\alpha \in \Omega^1(B)$  in the conformal class determined by the co-orientation on  $(B, \xi)$ ; and
- (3) A map of symplectic cones  $f : (\pi : M \rightarrow B, \omega) \rightarrow (\pi' : M' \rightarrow B, \omega)$  descends to a co-orientation preserving contact map  $\bar{f} : (B, \xi) \rightarrow (B', \xi')$ .

We provide a sketch of the proof:

*Proof.* Let  $\Xi$  be an expanding vector field for  $(M, \omega)$ . Then define  $\beta := \iota(\Xi)\omega$ . It follows that:

$$\xi_b := d\pi_m(\ker(\beta_m)) \text{ for any } m \in \pi^{-1}(b)$$

yields a well-defined distribution on  $B$ .

To see this is a co-orientable contact distribution, we turn to the second item above: for  $\varphi : M \rightarrow B \times \mathbb{R}$ , one may check that the form  $\alpha := (\varphi^* \beta)|_{B \times \{0\}}$  is a contact form for the distribution  $\xi$  and that  $\varphi^* \omega = d(e^t \alpha)$ .

To see  $\xi$  inherits a co-orientation, note that for any  $b \in B$ ,  $\pi^*$  yields a linear isomorphism from  $\xi_b^o$  to  $(\ker(\beta_m))_m^o$ . Hence, we may choose the co-orientation of  $\xi$  defining, for each  $b \in B$ ,

$$(\xi_+^o)_b := \{\eta \in \xi^o \setminus \{0\} \mid \pi^* \eta = \lambda \beta_m \text{ for some } \lambda > 0, m \in \pi^{-1}(b)\} \quad (\text{A.1})$$

Finally, note that, since a map of symplectic cones  $f : (M, \omega) \rightarrow (M', \omega')$  is  $\mathbb{R}$ -equivariant, it descends to a smooth map  $\bar{f} : B \rightarrow B'$  (for  $(B, \xi)$  and  $(B', \xi')$  the co-oriented contact manifolds associated to the  $\mathbb{R}$ -quotients  $M/\mathbb{R}$  and  $M'/\mathbb{R}$ ). Then for  $\Xi, \Xi'$  the expanding vector fields for  $(M, \omega), (M', \omega')$  respectively, we have that  $f_* \Xi = \Xi'$ . Then

$$f^*(\iota(\Xi')\omega') = f^*(\iota(f_* \Xi)\omega') = \iota(\Xi)f^* \omega' = \iota(\Xi)\omega$$

It follows from Equation (A.1) defining  $\xi$  and  $\xi'$  that  $\bar{f}$  is a co-orientation preserving contact map.  $\square$

It is not difficult to show that, if a symplectic cone  $(M, \omega)$  is, in fact a symplectic toric cone, its quotient inherits the structure of a contact toric manifold.

**Definition A.5.** A **contact toric manifold** is a co-oriented contact manifold  $(B, \xi)$  with an effective contact action by a torus  $G$  with dimension satisfying  $2 \dim(G) = \dim(B) + 1$ .

**Proposition A.6.** Let  $(M, \omega, \mu)$  be a symplectic toric manifold. Then the co-oriented contact manifold  $(B, \xi)$  with  $M/\mathbb{R} := B$  and  $\xi$  the contact distribution determined by  $(M, \omega)$  (as described in Proposition A.4) is a contact toric manifold.

In defining moment maps for contact manifolds, we first moment maps for any individual contact form.

**Definition A.7.** Let  $(B, \xi)$  be a co-oriented contact manifold with contact form  $\alpha$ . Suppose additionally that a Lie group  $G$  with Lie algebra  $\mathfrak{g}^*$  acts via contactomorphisms on  $(B, \xi)$  and the action preserves  $\alpha$ . Then the  $\alpha$  **moment map** for this action is the unique map  $\mu_\alpha : B \rightarrow \mathfrak{g}^*$  satisfying:

$$\langle \mu_\alpha(b), X \rangle = \alpha_b(X_B(b))$$

for every  $b \in B$ ,  $X \in \mathfrak{g}$  (here,  $X_B$  denotes the vector field on  $B$  induced by the action of  $X$ ). One may show  $\mu$  is equivariant (hence, for the torus, is  $G$ -invariant).

The above quotient process taking a symplectic toric manifold to a contact toric manifold  $\mathbb{R}$ -quotient has (a partial) inverse.

**Proposition A.8.** Let  $(B, \xi)$  be a co-oriented contact manifold acted on by Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . Then  $\xi_+^o$  inherits a  $G$  action, first by lifting the action to a symplectic action on  $T^*B$  and then restricting this action to  $\xi_+^o$ . Furthermore, this action is Hamiltonian. In general, the symplectic toric cone  $(B \times \mathbb{R}, d(e^t \alpha))$  also inherits the structure of a symplectic toric manifold.

It is important for the work above to find a  $G$ -equivariant trivialization of a symplectic cone with torus action commuting with the real action  $\mathbb{R}$ . This is always possible for the symplectization of contact toric manifolds, where this task is exactly the same as finding a  $G$ -invariant contact form.

**Proposition A.9.** Let  $(B, \xi)$  be a co-oriented contact manifold with an effective action of torus  $G$ . Then there exists a  $G$ -invariant contact form  $\alpha$  with  $\alpha$  serving as a section for  $\xi_+^o \rightarrow B$ .

$G$ -invariant  $\alpha$  in the above proposition is found by averaging any contact form against  $G$ . The proposition below shows how we can generalize this process.

**Proposition A.10.** Given a symplectic cone  $(\pi : M \rightarrow B, \omega)$  with effective action by torus  $G$  that commutes with the action of  $\mathbb{R}$ , there exists a trivialization of  $M$  as a principal  $\mathbb{R}$ -bundle  $\varphi : M \rightarrow B \times \mathbb{R}$  that is  $G$ -equivariant (where the contact action on  $B$  is trivially extended to an action on  $B \times \mathbb{R}$ ).

This fact is not standard, so here is a proof.

*Proof.* To start with, fix a global section of  $\pi$ :  $s : B \rightarrow M$  (this is always possible, as principal  $\mathbb{R}$ -bundles must always be trivial). Recall for any principal  $K$ -bundle  $\pi : P \rightarrow N$  for Lie group  $K$ , there exists a smooth “division map”  $d : P \times_N P \rightarrow K$ , the map uniquely defined by  $p \cdot d(p, p') = p'$ .

Then define the map  $f : B \times G \rightarrow \mathbb{R}$  by:

$$f(b, g) := d(s(g \cdot b), g \cdot s(b))$$

This is well-defined as  $\pi$  is  $G$ -equivariant. Essentially,  $f$  measures the failure of  $s$  to be  $G$ -equivariant and will be used to properly adjust  $s$  into an equivariant section.

$f$  satisfies the following useful property: for  $b \in B$ ,  $g, h \in G$ ,

$$\begin{aligned} f(h \cdot b, g) &= d(s(g \cdot (h \cdot b)), g \cdot s(h \cdot b)) \\ &= d(s(g \cdot (h \cdot b)), (gh) \cdot s(b)) + d(g \cdot (h \cdot s(b)), g \cdot s(h \cdot b)) \\ &= d(s(g \cdot (h \cdot b)), (gh) \cdot s(b)) + d((h \cdot s(b)), s(h \cdot b)) \\ &= f(b, gh) - f(b, h) \end{aligned} \tag{A.2}$$

Note the second line is equivalent to the third as  $d$  is  $G$ -invariant (with respect to the diagonal action of  $G$  on  $M \times_B M$ ); this follows since the actions of  $G$  and  $\mathbb{R}$  commute.

Now,  $f$  is averaged. Fix a  $G$ -invariant measure  $d\lambda$  on  $G$  with  $\int_G d\lambda = 1$ . Define  $\bar{f} : B \rightarrow \mathbb{R}$  by:

$$b \mapsto \int_G f(b, g) d\lambda$$

As  $\bar{f}$  is the result of integrating a smooth family of functions on  $G$  parameterized by  $B$ , it is smooth. Calculation (A.2) reveals that  $\bar{f}(h \cdot b) = \bar{f}(b) - f(b, h)$ .

Finally, define a new global section of  $M$   $\bar{s} : B \rightarrow M$  by  $\bar{s}(b) := s(b) \cdot (-\bar{f}(b))$ . Then:

$$\begin{aligned} \bar{s}(h \cdot b) &= s(h \cdot b) \cdot (-\bar{f}(h \cdot b)) \\ &= (s(h \cdot b) \cdot (f(b, h))) \cdot -\bar{f}(b) \\ &= h \cdot s(b) \cdot -\bar{f}(b) \\ &= h \cdot \bar{s}(b) \end{aligned}$$

The second and third lines are equivalent as, by definition,  $f(b, h)$  is the real number whose action takes  $s(h \cdot b)$  to  $h \cdot s(b)$ . So  $\bar{s}$  is an equivariant section. It follows that it defines a  $G$ -equivariant trivialization of  $M$  as a principal  $\mathbb{R}$ -bundle. □

Here is another important proposition we use in the paper.

**Proposition A.11.** Let  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  be a symplectic toric cone with homogeneous moment map  $\mu : M \rightarrow \mathfrak{g}^*$ . Then zero is not in the image of  $\mu$ .

*Proof.* Let  $(B, \xi)$  be the co-oriented contact toric manifold with  $M/\mathbb{R}$  as described above. As shown in Lemma 2.12 of [15], for  $\xi_+^o$  the symplectization of  $(B, \xi)$ , the image of the homogeneous moment map  $\nu : \xi_+^o \rightarrow \mathfrak{g}^*$  for  $\xi_+^o$  does not contain zero. Then, since every symplectic toric cone over  $(B, \xi)$  is  $(G \times \mathbb{R})$ -equivariantly to  $\xi_+^o$ , it follows that since  $\mu$  was also chosen to be homogeneous, any symplectomorphism between  $(M, \omega)$  and  $\xi_+^o$  must preserve moment maps. Thus, the image of  $\mu$  also does not contain zero. □

Following Lerman's approach in [15], we define a notion of the symplectic slice representation for contact toric manifolds.

**Definition A.12.** Let  $(B, \xi)$  be a co-oriented contact toric manifold with  $G$ -invariant contact form  $\alpha$ . Let  $\omega := (d\alpha)|_\xi$ . Then for any point  $x \in B$ , the  $\alpha$  symplectic slice representation at  $x$  is the  $G_x$ -vector space:

$$(V, \omega_V)_\alpha := \left( \frac{(T_x(G \cdot x) \cap \xi_x)^\omega}{T_x(G \cdot x) \cap \xi_x}, \omega|_V \right)$$

Note that another choice of  $G$ -invariant contact form  $\alpha' = e^f \alpha$  for  $(B, \xi)$  defines the same vector space  $V$  with symplectic form  $d(e^f \omega)$ . Thus, the symplectic vector space  $(V, \omega_V)_\alpha$  depends on a choice of contact form.

**Remark A.13.** This matches the definition of “symplectic slice representation” in Definition 3.8 of [15]. We choose to label this with the contact form  $\alpha$  defining this symplectic representation to avoid confusion with the standard symplectic slice representation for a symplectic toric manifold  $(M, \omega)$ : the vector space  $W := (T_x(G \cdot x))^\omega / T_x(G \cdot x)$  with symplectic form  $\omega_W := (\omega|_x)|_W$  and restricted action of  $G_x$ . We will use both in the following lemma.

**Lemma A.14.** Suppose  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  is a symplectic toric cone with quotient  $\pi : M \rightarrow B$  with base co-oriented contact toric manifold  $(B, \xi)$ . Then for each  $p \in M$ , there is a  $G$ -invariant contact form  $\alpha$  for  $(B, \xi)$  such that the  $\alpha$  symplectic slice representation  $(V, \omega_V)_\alpha$  of  $\pi(p)$  in  $(B, \xi)$  and the symplectic slice representation  $(W, \omega_W)$  of  $p$  in  $(M, \omega)$  are symplectically isomorphic as representations of  $G_{\pi(p)} = G_p$ .

*Proof.* Fix  $p \in M$  and define  $b := \pi(p)$ . Then, since the actions of  $\mathbb{R}$  and  $G$  on  $M$  commute, we have that  $G_p = G_b$ .

Let  $\varphi : M \rightarrow B \times \mathbb{R}$  be a  $G$ -equivariant trivialization of  $M$  as a principal  $\mathbb{R}$ -bundle such that  $\varphi(p) = (b, 0)$ . Then for  $\Xi$  the expanding vector field associated to  $(M, \omega)$ , define  $\alpha := (\varphi^*(\iota(\Xi)\omega))|_{B \times \{0\}}$ . We have that

$$d\alpha = d(\varphi^*(\iota(\Xi)\omega))|_{B \times \{0\}} = (\varphi^*d(\iota(\Xi)\omega))|_{B \times \{0\}} = (\varphi^*\omega)|_{B \times \{0\}}$$

So since  $\varphi$  is equivariant,  $d\varphi_{(b,0)}$  restricts to an isomorphism between  $T_{(b,0)}(G \cdot (b, 0))$ , and  $T_p(G \cdot p)$ . Thus, for  $v \in (T_b(G \cdot b) \cap \xi)^{d\alpha} \subset T_b B$  and  $w \in T_p(G \cdot p) \subset T_p M$ , there exists  $w' \in T_{(b,0)}(G \cdot (b, 0))$  with  $d\varphi_{(b,0)}(w') = w$  and we have that

$$\omega_p(d\varphi_{(b,0)}(v, 0), w) = \omega_{\varphi(p)}(d\varphi_{(b,0)}(v, 0), d\varphi_{(b,0)}(w')) = (\varphi^*\omega)_{(b,0)}((v, 0), w') = 0$$

It follows that  $d\varphi_{(b,0)}$  maps  $(T_b(G \cdot b) \cap \xi)^{d\alpha} \oplus \{0\} \subset T_b B \times T_0 \mathbb{R}$  to  $(T_p(G \cdot p))^\omega$ .

Now, note that, since  $\mu$  is a homogeneous moment map,  $\mu(p) \neq 0$ . Since

$$\omega(\Xi, X_M)(p) = (\iota_\Xi d\langle \mu, X \rangle)(p) = (L_\Xi d\langle \mu, X \rangle)(p) = \langle \mu(p), X \rangle$$

for every  $X \in \mathfrak{g}$ , we may then conclude there exists  $Y \in \mathfrak{g}$  with  $\omega(\Xi, Y_M)(p) \neq 0$ . Thus, we have a decomposition

$$T_p M = \ker((\iota(\Xi)\omega)_p) \oplus \mathbb{R} \cdot Y_M(p)$$

Furthermore,  $\ker(\iota(\Xi)\omega_p)$  decomposes into the sum  $U \oplus \mathbb{R} \cdot \Xi$  for  $U$  a subspace of  $\ker(\iota(\Xi)\omega_p)$ . It follows that any class  $[v]$  of the quotient  $(T_p(G \cdot p))^\omega / T_p(G \cdot p)$  has a representative  $v \in U$  and, for such a choice of  $v$ , there exists  $v' \in T_{\pi(b)} B$  with  $d\varphi_{(b,0)}(v', 0) = v$ . By Equation (A.1), we have that  $v' = d\pi_p(v) \in \xi_b$ .

We may finally conclude that  $d\varphi_{(b,0)}$  descends to an isomorphism between

$$(V, \omega_V)_\alpha := ((T_b(G \cdot b) \cap \xi)^{d\alpha|_\xi} / (T_b(G \cdot b)), d\alpha|_V)$$

and

$$(W, \omega_W) := (T_p(G \cdot p)^\omega / T_p(G \cdot p), \omega_W).$$

From above, we have this is a symplectic map and, since  $\varphi$  is equivariant, it follows this is also equivariant.  $\square$

We may now make use of the following lemma.

**Lemma A.15** (Lemma 3.9, [15]). Let  $(B, \xi)$  and  $(B', \xi')$  be two co-oriented contact toric manifolds with  $G$ -invariant contact forms  $\ker \alpha = \xi$  and  $\ker \alpha' = \xi'$ . Suppose  $x \in B$  and  $x' \in B'$  satisfy

- $\Psi_\alpha(x) = \lambda \Psi_{\alpha'}(x')$  for  $\Psi_\alpha, \Psi_{\alpha'}$  the moment maps for  $\alpha, \alpha'$  and  $\lambda > 0$ ;
- $G_x = G_{x'}$  (i.e., the isotropy groups for each point are equal); and
- For  $(V, \omega)_\alpha$  and  $(V', \omega_{V'})_{\alpha'}$  the  $\alpha/\alpha'$  symplectic slice representations for  $x/x'$ , there is an  $G_x$ -equivariant linear isomorphism  $l : V \rightarrow V'$  such that  $l^* \omega_{V'} = (d(e^g \alpha)_x)|_V$  for some function  $g \in C^\infty(B)$

Then there are  $G$ -invariant open neighborhoods  $U$  of  $x$  and  $U'$  of  $x'$  and a  $G$ -equivariant diffeomorphism  $\varphi : U \rightarrow U'$  satisfying  $\varphi(x) = \varphi'(x')$  and  $\varphi^* \alpha' = f \alpha$  for some  $f \in C^\infty(U)$ .

This allows us to prove the following extension of a standard symplectic toric result to symplectic toric cones.

**Proposition A.16.** Let  $(M, \omega, \mu : M \rightarrow \mathfrak{g}^*)$  and  $(M', \omega', \mu' : M' \rightarrow \mathfrak{g}^*)$  be two symplectic toric cones. Suppose two points  $p \in M$  and  $p' \in M'$  have the same stabilizers so that the symplectic slice representations  $(V, \omega_V)$  and  $(V', \omega_{V'})$  are isomorphic as symplectic  $G_p = G_{p'}$  vector spaces and  $\mu(p) = \mu'(p')$ . Then there exist  $(G \times \mathbb{R})$ -invariant neighborhoods  $U$  and  $U'$  of  $p$  and  $p'$  respectively and a  $(G \times \mathbb{R})$ -equivariant symplectomorphism  $f : U \rightarrow U'$  with  $f(p) = f(p')$  and  $\mu'|_U = \mu|_U$ .

*Proof.* Let  $(B, \xi)$  and  $(B', \xi')$  be the contact toric bases of  $(M, \omega)$  and  $(M', \omega')$ . Denote the  $\mathbb{R}$ -quotient maps  $\pi : M \rightarrow B$  and  $\pi' : M' \rightarrow B'$  and define  $b := \pi(p)$  and  $b' := \pi'(p')$ . Then by Lemma A.14, there exist trivializations  $\varphi : B \times \mathbb{R} \rightarrow M$  and  $\varphi' : B' \times \mathbb{R} \rightarrow M'$  so that  $\varphi(b, 0) = p$  and  $\varphi'(b', 0) = p'$  and, for  $\varphi^* \omega = d(e^t \alpha)$  and  $\varphi'^* \omega' = d(e^t \alpha')$ , the symplectic slice representations  $(V, \omega_V)$  and  $(V', \omega_{V'})$  are isomorphic to the  $\alpha$  and  $\alpha'$  symplectic slice representations  $(W, \omega_W)_\alpha$  and  $(W', \omega_{W'})_{\alpha'}$  of  $b$  and  $b'$ , respectively.

Then by Lemma A.15 there are  $G$ -invariant neighborhoods of  $b$  and  $b'$  and a  $G$ -equivariant co-orientation preserving contactomorphism  $\phi : U \rightarrow U'$  with  $\phi(b) = b'$  and  $\phi^* \alpha' = e^g \alpha$ , for some  $g \in C^\infty(B)$ . The map  $\tilde{\phi} : U \times \mathbb{R} \rightarrow U' \times \mathbb{R}$ , defined by  $\tilde{\phi}(b, t) := (\phi(b), t - g(b))$  is  $(G \times \mathbb{R})$ -equivariant and satisfies  $\tilde{\phi}^*(d(e^t \alpha')) = d(e^t \alpha)$ . Hence,  $f := \varphi' \circ \tilde{\phi} \circ \varphi^{-1}$  yields a map of symplectic toric cones between  $\pi^{-1}(U)$  and  $\pi'^{-1}(U')$ . Since  $\mu|_U$  and  $\mu' \circ f$  are both homogeneous moment maps for  $\pi^{-1}(U)$ , it follows that  $\mu|_U = \mu' \circ f$ .

Finally, note that

$$f(p) = \varphi'(\tilde{\phi}(\varphi^{-1}(p))) = \varphi'(\tilde{\phi}(b, 0)) = \varphi'(\phi(b), -g(b)) = \varphi'(b', -g(b)).$$

Thus,  $f(p) = -g(b) \cdot p'$ . But then

$$\mu(p) = \mu'(f(p)) = \mu'(-g(b) \cdot p') = e^{-g(b)} \mu'(p'),$$

so, since  $\mu(p) = \mu'(p')$ , we must conclude that  $g(b) = 0$  (as the image of  $\mu'$  may not contain 0) and therefore  $f(p) = p'$ .  $\square$

Finally, we may prove Lemma 5.5; for convenience, we recall its statement:

**Lemma 5.5.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding and let  $q : W \rightarrow W/\mathbb{R}$  be the quotient by the natural  $\mathbb{R}$  action induced by  $\psi$ . Then any two symplectic toric cones over  $\psi$   $(M, \omega, \pi : M \rightarrow W)$  and  $(M', \omega', \pi' : M' \rightarrow W)$  are locally isomorphic; explicitly, there is an open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $W/\mathbb{R}$  and a collection of isomorphisms

$$\{\varphi_\alpha : (M, \omega, \pi : M \rightarrow W)|_{U_\alpha} \rightarrow (M', \omega', \pi' : M' \rightarrow W)|_{U_\alpha} \in \text{STC}_\psi(U_\alpha) \mid \alpha \in A\}.$$

*Proof.* With Proposition A.16, we may proceed exactly as in Lemma B.4 of [13]. Fix  $p \in M$  and  $p' \in M'$  with  $\pi(p) = \pi'(p')$ . Since  $\psi \circ \pi$  and  $\psi \circ \pi'$  are moment maps for  $(M, \omega)$  and  $(M', \omega')$ , respectively, we may conclude from the local normal form for symplectic toric manifolds that both  $p$  and  $p'$  have stabilizer  $K_{\pi(p)}$  and symplectic slice representations isomorphic to  $\mathbb{C}^k$  with symplectic weights  $\{v_1^*, \dots, v_k^*\}$ , for  $C_{\psi(\pi(p)), \{v_1^*, \dots, v_k^*\}}$  the unimodular cone uniquely determined by  $\psi$  near  $\pi(p)$ .

Therefore, by applying Proposition A.16, we have our result.  $\square$

Again following Lerman in [15], orbits in contact toric manifolds have a local normal form (as we are interested only in the structure of the neighborhood as a  $G$ -manifold, we suppress the additional information from the lemma regarding contact structure and moment map).

**Lemma A.17** (Lemma 3.10, [15]). Let  $(L, \xi)$  be a contact toric manifold with  $G$ -invariant form  $\alpha$  and  $\alpha$  moment map  $\Psi_\alpha$ . Given point  $p \in L$ , denote the  $\alpha$  symplectic slice representation by  $G_p \rightarrow Sp(V, \omega_V)$  and let  $\mathfrak{k} := (\mathbb{R}\Psi_\alpha(p))^o$  (the so-called *characteristic subalgebra* of the embedding  $G \cdot p \rightarrow (M, \xi)$ ). Then there exists a  $G$ -invariant neighborhood of the orbit of  $p$  in  $L$  that is  $G$ -equivariantly diffeomorphic to a neighborhood of the zero section of the vector bundle  $N = G \times_{G_p} ((\mathfrak{g}/\mathfrak{k})^* \oplus V)$

The above local normal form allows us to prove the following Lemma that will be important to us.

**Lemma A.18.** Let  $(B, \xi)$  be a contact toric manifold. Then the quotient  $B/G$  is a manifold with corners.

*Proof.* For each point  $p \in B$ , the stabilizer  $G_p$  is a torus (see Lemma 3.13 of [15]). Let  $\alpha$  be a  $G$ -invariant contact form for  $(B, \xi)$ . From the above lemma, there exists subspace  $U \subset \mathfrak{g}^*$  so that, for  $\alpha$  symplectic slice representation  $G_p \rightarrow Sp(V, \omega_V)$ , there is a  $G$ -invariant neighborhood of  $G \cdot p$  in  $B$  that is  $G$ -equivariantly diffeomorphic to a neighborhood of the zero section of the vector bundle:  $N = G \times_{G_p} (U \times V)$  (where  $U$  has trivial  $G_p$  action).

So to understand what  $B/G$  locally looks like, it is enough to understand  $N/G$ . Well:

$$\begin{aligned} N/G &= ((G \times U \times V)/G_p)/G \\ &= ((G \times U \times V)/G)/G_p \\ &= U \times V/G_p \end{aligned}$$

Here, we may reverse the quotients as the actions of  $G$  and  $G_p$  commute. Since  $G_p$  is a torus, we can decompose  $V$  into weight spaces (as in the appendix of [17]) and easily see  $V/G_p$  is a manifold with corners, diffeomorphic to a sector (i.e., a manifold with corners of



the form  $[0, \infty)^k \times \mathbb{R}^l$ ). So the  $G$ -equivariant diffeomorphism above descends to a manifold with corners chart for  $[p] \in B/G$  centered at the origin in  $U \times V/G_p$ .  $\square$

## APPENDIX B. STACKS

In this appendix, we will provide some notes on stacks. This will also contain proofs that our presheaves of groupoids  $\mathbf{HSTB}_\psi$  and  $\mathbf{CSTB}_\psi$  are stacks as well as the major technical lemma we require to prove that  $\mathbf{hc} : \mathbf{HSTB}_\psi \rightarrow \mathbf{STC}_\psi$  and  $\tilde{\mathbf{c}} : \mathbf{CSTB}_\psi \rightarrow \mathbf{STSS}_\psi$  are isomorphisms of presheaves of groupoids.

For simplicity's sake, we will be using a less general definition that would perhaps be more accurately named a sheaf of groupoids. Since the stacks we are interested in are, in fact, arising from presheaves of groupoids, this will be ideal in our case (rather than using lax presheaves or categories fibered in groupoids). Additionally, rather than using Grothendieck topologies to define stacks over categories, for our purposes we need only worry about defining stacks over categories of open subsets of a topological space (or full subcategories of these categories), as in the case of traditional sheaves of sets. A few good sources for the complete story on stacks are [24] (which is focused more on stacks in algebraic geometry), [2] (which is focused on using stacks in differential geometry), and [11] (which discusses stacks over manifolds and over topological spaces).

Fix a topological space  $X$ . For  $\{U_\alpha\}_{\alpha \in A}$  an open cover of some topological space  $Y$ , we write  $U_{\alpha\beta} := U_\alpha \cap U_\beta$  and  $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ . First, we need some preliminaries.

**Definition B.1.**  $\mathbf{Open}(X)$  is the category of open sets of  $X$ : the objects of  $X$  are open subsets  $U \subset X$  and the morphisms are inclusions of open subsets  $\iota : U \rightarrow V$ . Write  $\mathbf{Open}(X)^{op}$  for the opposite category of  $\mathbf{Open}(X)$ .

Now we may define the category of descent data for a presheaf of groupoids:

**Definition B.2.** Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $X$  and let  $\mathcal{F} : \mathbf{Open}(X)^{op} \rightarrow \mathbf{Groupoids}$  be a presheaf of groupoids. Then  $\{U_\alpha\}_{\alpha \in A}$  descent data for  $\mathcal{F}$  are pairs of tuples

$$(\{\xi_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in A}, \{\varphi_{\alpha\beta} : \xi_\alpha|_{U_{\alpha\beta}} \rightarrow \xi_\beta|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A})$$

such that the morphisms  $\{\varphi_{\alpha\beta} : \xi_\alpha|_{U_{\alpha\beta}} \rightarrow \xi_\beta|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A}$  (known as **transition morphisms**) satisfy the **cocycle condition**: for every non-empty triple intersection  $U_{\alpha\beta\gamma}$ , we have that  $\varphi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \circ \varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = \varphi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$ .

A morphism of descent data

$$\{\eta_\alpha\}_{\alpha \in A} : (\{\xi_\alpha\}_{\alpha \in A}, \{\varphi_{\alpha\beta}\}_{\alpha, \beta \in A}) \rightarrow (\{\xi'_\alpha\}_{\alpha \in A}, \{\varphi'_{\alpha\beta}\}_{\alpha, \beta \in A})$$

is a collection of morphisms  $\{\eta_\alpha : \xi_\alpha \rightarrow \xi'_\alpha\}_{\alpha \in A}$  so that the diagram

$$\begin{array}{ccc} \xi_\alpha & \xrightarrow{\eta_\alpha} & \xi'_\alpha \\ \varphi_{\alpha\beta} \downarrow & & \downarrow \varphi'_{\alpha\beta} \\ \xi_\beta & \xrightarrow{\eta_\beta} & \xi'_\beta \end{array} \tag{B.1}$$

commutes for every  $\alpha$  and  $\beta$  with  $U_{\alpha\beta}$  non-empty.

Write  $\mathcal{D}_\mathcal{F}(\{U_\alpha\}_{\alpha \in A})$  for the **descent category**: the category of  $\{U_\alpha\}_{\alpha \in A}$  descent data for  $\mathcal{F}$  with morphisms of descent data.

The idea behind a stack is that the process of restricting any piece of global data to a piece of descent data, which may be thought of as a functor, is in fact an equivalence of categories. Here is a formal definition.

**Definition B.3.** Let  $\mathcal{F} : \text{Open}(X)^{op} \rightarrow \text{Groupoids}$  be a presheaf of groupoids. For an open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $X$ , define the restriction functor  $\Phi : \mathcal{F}(X) \rightarrow \mathcal{D}_F(\{U_\alpha\})$  as the functor taking an object  $\xi \in \mathcal{F}(X)$  to the descent data:

$$(\{\xi|_{U_\alpha}\}_{\alpha \in A}, \{id : (\xi|_{U_\alpha})|_{U_{\alpha\beta}} \rightarrow (\xi|_{U_\beta})|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A})$$

and a morphism  $\varphi : \xi \rightarrow \xi'$  to the morphism of descent data  $\{\varphi|_{U_\alpha} : \xi|_{U_\alpha} \rightarrow \xi'|_{U_\alpha}\}_{\alpha \in A}$ .

Then  $\mathcal{F}$  is a **stack** if, for every open subset  $U$  of  $X$  and for every open cover  $\{U_\alpha\}_{\alpha \in A}$ , the restriction morphism  $\Phi : \mathcal{F}_U(U) \rightarrow \mathcal{D}_{\mathcal{F}_U}(\{U_\alpha\}_{\alpha \in A})$  is an equivalence of groupoids.

**Definition B.4.** For  $\mathcal{F} : \text{Open}(X)^{op} \rightarrow \text{Groupoids}$  a presheaf and  $U$  any open subset of  $X$ , note that, for  $V \subset U$  an open subset, the restriction morphism from  $U$  to  $V$  is a map of groupoids. Thus, for any two objects  $\xi$  and  $\xi'$  in  $\mathcal{F}(U)$ , we have a map of sets

$$\text{Hom}_{\mathcal{F}(U)}(\xi, \xi') \rightarrow \text{Hom}_{\mathcal{F}(V)}(\xi|_V, \xi'|_V)$$

It is easy then to check that this corresponds to a presheaf of sets we will write as  $\underline{\text{Hom}}(\xi, \xi') : \text{Open}(U)^{op} \rightarrow \text{Sets}$ .

Say the presheaf  $\mathcal{F}$  is a **prestack** if for every open subset  $U \subset X$  and any two  $\xi$  and  $\xi'$  in  $\mathcal{F}(U)$ , the presheaf  $\underline{\text{Hom}}(\xi, \xi') : \text{Open}(U)^{op} \rightarrow \text{Sets}$  is a sheaf of sets.

**Remark B.5.** It is more or less clear that, for  $W$  a manifold with an  $\mathbb{R}$  action, we may just as easily define a stack over the category  $\text{Open}_{\mathbb{R}}(W)$ , as defined in Definition 3.10. Indeed, a presheaf of groupoids over this category is again just a functor  $\mathcal{F} : \text{Open}_{\mathbb{R}}(W)^{op} \rightarrow \text{Groupoids}$  and we may replace the open covers of  $\text{Open}(W)$  as in Definition B.3 with open covers of elements of  $\text{Open}_{\mathbb{R}}(W)$  by  $\mathbb{R}$ -invariant subsets.

**Remark B.6.** It is easy to check that a presheaf of groupoids  $\mathcal{F} : \text{Open}(U)^{op} \rightarrow \text{Groupoids}$  is a prestack if and only if for every open subset  $U \subset X$  and for any open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $U$  the restriction functor  $\Phi : \mathcal{F} \rightarrow \mathcal{D}_{\mathcal{F}_U}(\{U_\alpha\})$  is fully faithful.

Note also that all the presheaves of groupoids we consider in this paper are clearly prestacks. Since the groupoids of these presheaves consist of spaces with extra information (bundles/manifolds/stratified spaces with symplectic forms) and the morphisms are maps of these spaces, it is clear that, in these cases, collections of maps between two objects on local restrictions that are coherent will glue to a unique map.

**Example B.7.** Fix a topological space  $X$  and let  $\text{BG} : \text{Open}(X)^{op} \rightarrow \text{Groupoids}$  be the presheaf of principal  $G$ -bundles over  $X$ : for each open  $U$ , the groupoid  $\text{BG}(U)$  is that with objects principal  $G$ -bundles over  $U$  and with morphisms isomorphisms of principal  $G$ -bundles ( $G$ -equivariant maps of bundles covering the identity on  $U$ ). The restriction morphisms  $\text{BG}(V) \rightarrow \text{BG}(U)$  for  $U \subset V$  open subsets of  $X$  are simply just the morphisms taking principal bundles  $\pi : P \rightarrow V$  to  $\pi|_U : P|_U \rightarrow U$ .

The proof that  $\text{BG}$  is a stack comes in two easy parts. Let  $U$  be an open subset of  $X$  with open cover  $\{U_\alpha\}_{\alpha \in A}$  and let  $\Phi : \text{BG}(U) \rightarrow \mathcal{D}_{\text{BG}}(\{U_\alpha\}_{\alpha \in A})$  be the restriction functor. Let  $\pi : P \rightarrow U$  and  $\pi' : P' \rightarrow U$  be two principal  $G$ -bundles. Then it is clear that  $\underline{\text{Hom}}(P, P')$  is a sheaf of sets. Thus,  $\Phi$  is fully faithful (see Remark B.6).

To show  $\Phi$  is essentially surjective, let

$$(\{\pi_\alpha : P_\alpha \rightarrow U_\alpha\}_{\alpha \in A}, \{\varphi_{\alpha\beta} : P_\alpha \rightarrow P_\beta\}_{\alpha, \beta \in A})$$

be a piece of decent data. Then the construction

$$P := \left( \bigsqcup_{\alpha \in A} P_\alpha \right) / \sim$$

where  $\sim$  is the equivalence relation generated by

$$p \sim q \text{ if } p \in P_\alpha|_{U_{\alpha\beta}}, \ q \in P_\beta|_{U_{\alpha\beta}}, \text{ and } \varphi_{\alpha\beta}(p) = q$$

yields the total space of a principal  $G$ -bundle with quotient map  $\pi : P \rightarrow U$  for  $\pi([p]) = \pi_\alpha(p)$ , where  $p \in P_\alpha$ . It is clear then that  $\Phi(P)$  is isomorphic to our original descent data.

In the case where  $X$  is a manifold with corners, it follows as above that the presheaf of principal  $G$ -bundles of manifolds with corners is also a stack.

**Proposition B.8.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a homogeneous unimodular local embedding. Then  $\text{HSTB}_\psi : \text{Open}_\mathbb{R}(W)^{op} \rightarrow \text{Groupoids}$  is a stack.

*Proof.* Recall  $\text{HSTB}_\psi$  is a presheaf of groupoids over  $W$  with an  $\mathbb{R}$ -invariant open subset  $U \subset W$  corresponding to the groupoid  $\text{HSTB}_{\psi|_U}(U)$ . Fix an open cover of  $U$  by  $\mathbb{R}$ -invariant subsets  $\{U_\alpha\}_{\alpha \in A}$ . Then we must show  $\Phi : \text{HSTB}_\psi(U) \rightarrow \mathcal{D}_{\text{HSTB}_\psi}(\{U_\alpha\})$  is an equivalence of categories.

Let  $(\pi : P \rightarrow U, \omega)$  and  $(\pi' : P' \rightarrow U, \omega')$  be any two homogeneous symplectic toric bundles in  $\text{HSTB}_\psi(U)$ . Then as morphisms in  $\text{HSTB}_\psi$  must be  $(G \times \mathbb{R})$ -equivariant symplectomorphisms, it follows that any family of morphisms  $\varphi_\alpha : P|_{U_\alpha} \rightarrow P'|_{U_\alpha}$  in  $\text{HSTB}_\psi$  that successfully patch together to a map of principal  $G$ -bundles must also patch together to an  $\mathbb{R}$ -equivariant symplectomorphism. Thus,  $\underline{\text{Hom}}((\pi : P \rightarrow U, \omega), (\pi' : P' \rightarrow U, \omega'))$  must be a sheaf and therefore  $\Phi$  is fully faithful (see Remark B.6).

The case of principal  $G$ -bundles provides a guide for showing  $\Phi$  is essentially surjective. Let

$$((\pi_\alpha : P_\alpha \rightarrow U_\alpha, \omega_\alpha)_{\alpha \in A}, \{\varphi_{\alpha\beta} : (P_\alpha, \omega_\alpha)|_{U_{\alpha\beta}} \rightarrow (P_\beta, \omega_\beta)|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A}) \quad (\text{B.2})$$

be a piece of descent data. As in the case of principal  $G$ -bundles, let  $\pi : P \rightarrow U$  be the bundle built as in Example B.7. Since the transition maps  $\varphi_{\alpha\beta}$  are  $\mathbb{R}$ -equivariant, it follows that the actions of  $\mathbb{R}$  on each  $P_\alpha$  patch together to give a free and proper action on  $P$ .

As the transition maps  $\varphi_{\alpha\beta}$  must also be symplectomorphisms, it is clear that the symplectic forms from each piece must patch together. Finally, since the condition  $\rho_\lambda^* \omega = e^\lambda \omega$  for  $\rho_\lambda : P \rightarrow P$  the action diffeomorphism for real  $\lambda$  is local, it follows that, since each  $\omega_\alpha$  satisfies this property,  $\omega$  must satisfy this property as well. So descent data correctly patches together to an element  $(\pi : P \rightarrow U, \omega)$  with  $\Phi(\pi : P \rightarrow U, \omega)$  isomorphic to descent data (B.2).

Thus,  $\text{HSTB}_\psi$  is a stack. □

**Proposition B.9.** Let  $\psi : W \rightarrow \mathfrak{g}^*$  be a stratified unimodular local embedding. Then  $\text{CSTB}_\psi : \text{Open}(W)^{op} \rightarrow \text{Groupoids}$  is a stack.

*Proof.* Fix an open subset  $U$  in  $W$  with open cover  $\{U_\alpha\}_{\alpha \in A}$ . We must show  $\Phi : \text{CSTB}_\psi(U) \rightarrow \mathcal{D}_{\text{CSTB}_\psi}(\{U_\alpha\})$  is an equivalence of categories.

Let  $(\pi : P \rightarrow U_{\text{reg}}, \omega)$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')$  be any two conical symplectic toric bundles in  $\text{CSTB}_\psi(U)$ . Then since maps of conical symplectic toric bundles between  $(\pi : P \rightarrow U_{\text{reg}}, \omega)$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')$  are isomorphisms of principal  $G$ -bundles that are also symplectomorphisms, it follows easily that, since checking a map is symplectic may be done locally, that coherent families of maps between  $(\pi : P \rightarrow U_{\text{reg}}, \omega)|_{U_\alpha}$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')|_{U_\alpha}$  for each  $\alpha$  must glue to a map between  $(\pi : P \rightarrow U_{\text{reg}}, \omega)$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')$ . On the other hand, as a map between  $(\pi : P \rightarrow U_{\text{reg}}, \omega)$  and  $(\pi' : P' \rightarrow U_{\text{reg}}, \omega')$  is uniquely determined by its collection of restrictions, it is clear

$$\underline{\text{Hom}}((\pi : P \rightarrow U, \omega), (\pi' : P' \rightarrow U, \omega'))$$

must be a sheaf and therefore  $\Phi$  is fully faithful (see Remark B.6).

Again, to show  $\Phi$  is an essentially surjective functor, we use  $\text{BG}$  as a model. Let

$$(\{(\pi_\alpha : P_\alpha \rightarrow U_{\alpha\text{reg}}, \omega_\alpha)\}_{\alpha \in A}, \{\varphi_{\alpha\beta} : (P_\alpha, \omega_\alpha)|_{U_{\alpha\beta}} \rightarrow (P_\beta, \omega_\beta)|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A})$$

be a piece of descent data. Let  $\pi : P \rightarrow U$  be the principal  $G$ -bundle  $\pi : P \rightarrow U$  built from the bundles of the above descent data as in Example B.7. As isomorphisms in  $\text{CSTB}_\psi(U_{\alpha\beta})$  must, in particular, be symplectomorphisms, it follows that the symplectic forms  $\{\omega_\alpha\}_{\alpha \in A}$  patch together to a symplectic form  $\omega$  on  $P$ . As  $\psi \circ \pi_\alpha$  is a moment map for each  $\omega_\alpha$  and, since  $\psi \circ \pi_\alpha = \psi \circ \pi|_{U_\alpha}$  and  $\omega|_{U_\alpha} = \omega_\alpha$ , it follows the glued map  $\pi$  must be a moment map for  $\omega$ .

Finally, for each singularity  $w$  of  $W$ , and for any element  $U_\alpha$  of the cover  $\{U_\alpha\}_{\alpha \in A}$  containing  $w$ , by definition there must be an open subset  $V \subset U_w$  containing  $w$  so that  $(P|_{U_\alpha}, \omega_\alpha)|_V$  is isomorphic to a neighborhood of  $-\infty$  in a symplectic cone. It then follows that  $(P, \omega)|_V$  is also isomorphic to a neighborhood of  $-\infty$  in a symplectic cone. Therefore,  $(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is an element of  $\text{CSTB}_\psi(U)$ . It is clear then that  $\Phi(\pi : P \rightarrow W_{\text{reg}}, \omega)$  is isomorphic to the above descent data. Thus  $\Phi$  is essentially surjective and  $\text{CSTB}_\psi$  is a stack.  $\square$

A special class of presheaves we are interested in are so-called transitive stacks.

**Definition B.10.** A presheaf of groupoids  $\mathcal{F} : \text{Open}(X)^{\text{op}} \rightarrow \text{Groupoids}$  is called **transitive** if, for every open subset  $U \subset X$ , any two objects  $\xi$  and  $\xi'$  in  $\mathcal{F}(U)$  are locally isomorphic; that is, there exists a cover  $\{U_\alpha\}_{\alpha \in A}$  such that the restrictions  $\xi|_{U_\alpha}$  and  $\xi'|_{U_\alpha}$  are isomorphic for each  $\alpha$ .

The payoff for working with stacks in our case will be the following technical lemma. This is a generalized version of the proof presented in [13] that, for  $\psi : W \rightarrow \mathfrak{g}^*$  a unimodular local embedding, the functor  $c|_U : \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$  is essentially surjective on each open subset  $U \subset W$ .

**Lemma B.11.** Let  $X$  be a topological space. Suppose  $\mathcal{F} : \text{Open}(X)^{\text{op}} \rightarrow \text{Groupoids}$  is a stack and that  $\mathcal{G} : \text{Open}(X)^{\text{op}} \rightarrow \text{Groupoids}$  is a prestack. Then, if for each open set  $U$ , a map of presheaves  $\Psi : \mathcal{F} \rightarrow \mathcal{G}$  satisfies

- (1)  $\Psi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is fully faithful
- (2) For each  $x \in U$  and each  $\xi \in \mathcal{G}(U)$ , there is an open subset  $V \subset U$  and an element  $\eta \in \mathcal{F}(V)$  such that  $\Psi(\eta)$  is isomorphic to  $\xi|_V$

$\Psi$  must be an isomorphism of presheaves. Thus,  $\mathcal{G}$  must be a stack and  $\Psi$  is in fact an isomorphism of sheaves.

**Remark B.12.** Note that, in the case of the map of presheaves  $\mathbf{hc} : \mathbf{HSTB}_\psi \rightarrow \mathbf{STC}_\psi$  over  $\mathbf{Open}_{\mathbb{R}}(W)$ , this lemma still works (if we are sure to use open  $\mathbb{R}$ -invariant subsets and covers by open  $\mathbb{R}$ -invariant subsets of  $W$ ). Additionally, if  $\mathcal{G}$  is a transitive prestack and, for every open  $U$ ,  $\mathcal{F}(U)$  is non-empty, any map of presheaves satisfies condition (2) of the above lemma; in fact, elements of  $\mathcal{G}(U)$  are always locally isomorphic to any elements of the image of  $\mathcal{F}(U)$ . This will also be the case with  $\mathbf{hc} : \mathbf{HSTB}_\psi \rightarrow \mathbf{STC}_\psi$ . However, we must also have this slightly more general version of the lemma to apply to the case of  $\tilde{\mathbf{c}} : \mathbf{CSTB}_\psi \rightarrow \mathbf{STSS}_\psi$ , where  $\mathbf{STSS}_\psi$  in general *need not be* a transitive prestack.

*Proof of Lemma B.11.* Fix an open subset  $U$  of  $X$ . To show  $\Psi$  is an isomorphism of presheaves, it is enough to show that  $\Psi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is an equivalence of groupoids for each  $U$ . By hypothesis, we have already that  $\Psi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is fully faithful, so it remains to show that it is essentially surjective.

Fix an element  $\xi \in \mathcal{G}(U)$ . Then by hypothesis there is an open cover  $\{U_\alpha\}_{\alpha \in A}$  of  $U$ , elements  $\{\eta_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in A}$ , and a family of isomorphisms  $\{\varphi_\alpha : \Psi(\eta_\alpha) \rightarrow \xi|_{U_\alpha}\}_{\alpha \in A}$ . Then, since  $\Psi_{U_{\alpha\beta}}$  is full for every  $U_{\alpha\beta}$  and since  $(\Psi(\eta_\alpha))|_{U_{\alpha\beta}} = \Psi(\eta_\alpha|_{U_{\alpha\beta}})$ , there exist morphisms

$$\phi_{\alpha\beta} : \eta_\alpha|_{U_{\alpha\beta}} \rightarrow \eta_\beta|_{U_{\alpha\beta}}$$

for every  $\alpha$  and  $\beta$  with  $U_{\alpha\beta}$  non empty such that  $\Psi(\phi_{\alpha\beta}) = \varphi_\beta^{-1} \varphi_\alpha$ .

For any  $\alpha$ ,  $\beta$ , and  $\gamma$  so that  $U_{\alpha\beta\gamma}$  is non-empty, as  $\Psi_{U_{\alpha\beta\gamma}}$  is faithful, it follows that  $\phi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \circ \phi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = \phi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$ . Thus, the family of isomorphisms  $\{\phi_{\alpha\beta}\}_{\alpha, \beta \in A}$  satisfies the cycle condition.

Therefore, the pair of subsets  $\{\{\eta_\alpha\}_{\alpha \in A}, \{\phi_{\alpha\beta}\}_{\alpha, \beta \in A}\}$  is a piece of descent data for  $\mathcal{F}$  with respect to the cover  $\{U_\alpha\}_{\alpha \in A}$ . As  $\mathcal{F}$  is a stack, the restriction functor  $\Phi : \mathcal{F}(U) \rightarrow \mathcal{D}_{\mathcal{F}}(\{U_\alpha\}_{\alpha \in A})$  from  $\mathcal{F}(U)$  to the descent category is essentially surjective. Thus, there exists an element  $\eta$  in  $\mathcal{F}(U)$  and an isomorphism of descent data:

$$\{\rho_\alpha\}_{\alpha \in A} : \Phi(\eta) \rightarrow \{\{\eta_\alpha\}_{\alpha \in A}, \{\phi_{\alpha\beta} : \eta_\alpha|_{U_{\alpha\beta}} \rightarrow \eta_\beta|_{U_{\alpha\beta}}\}_{\alpha, \beta \in A}\}$$

Finally, we have the commutative diagram

$$\begin{array}{ccccc} \Psi(\eta)|_{U_{\alpha\beta}} & \xrightarrow{\Psi(\rho_\alpha)|_{U_{\alpha\beta}}} & \Psi(\eta_\alpha)|_{U_{\alpha\beta}} & \xrightarrow{\varphi_\alpha} & \xi|_{U_{\alpha\beta}} \\ \parallel & & \downarrow \Psi(\phi_{\alpha\beta}) & & \parallel \\ \Psi(\eta)|_{U_{\alpha\beta}} & \xrightarrow{\Psi(\rho_\beta)|_{U_{\alpha\beta}}} & \Psi(\eta_\beta)|_{U_{\alpha\beta}} & \xrightarrow{\varphi_\beta} & \xi|_{U_{\alpha\beta}} \end{array}$$

for every  $\alpha$  and  $\beta$  with  $U_{\alpha\beta}$  non-empty. This commutes as the left square is exactly the image under  $\Psi$  of the diagram (B.1) corresponding to the isomorphism of descent data  $\{\rho_\alpha\}_{\alpha \in A}$  while the right hand side commutes by definition of  $\phi_{\alpha\beta}$ .

For each  $\alpha$ , let  $f_\alpha : \Psi(\eta)|_{U_\alpha} \rightarrow \xi|_{U_\alpha}$  be the composition  $f_\alpha : \varphi_\alpha \circ \Psi(\rho_\alpha)|_{U_\alpha}$ . Then the above diagram demonstrates that, on the intersections  $U_{\alpha\beta}$ , the  $f_\alpha$ 's are coherent. As  $\mathcal{G}$  is a prestack,  $\underline{\mathbf{Hom}}(\Psi(\eta), \xi)$  is a sheaf and the isomorphisms  $\{f_\alpha\}_{\alpha \in A}$  glue to an isomorphism  $f : \Psi(\eta) \rightarrow \xi$ .

Thus,  $\Psi_U : \mathcal{F} \rightarrow \mathcal{G}$  is essentially surjective for every  $U$  and, by hypothesis, is fully faithful. Therefore,  $\Psi$  is an isomorphism of presheaves.  $\square$

## APPENDIX C. RELATIVE DE RHAM COHOMOLOGY

In this section, we review relative de Rham cohomology, as presented by Bott and Tu [3]. While their treatment uses manifolds, it should be more or less clear that, as we are not using any special properties of manifolds beyond the existence of the de Rham complex, everything generalizes to manifolds with corners.

**Definition C.1.** Let  $M$  and  $N$  be two manifolds with corners and  $f : M \rightarrow N$  a smooth map. Then the relative de Rham complex  $\Omega^\bullet(f)$  is the cochain complex with  $\Omega^p(f) := \Omega^p(N) \oplus \Omega^{p-1}(M)$  and differential  $d_f((\alpha, \beta)) := (d\alpha, f^*\alpha - d\beta)$  (here, we take  $\Omega^k(M) = 0$  for  $k < 0$  and  $\Omega^0(f) = C^\infty(N)$ ). Denote by  $H^\bullet(f)$  the cohomology of this cochain complex.

In the case where  $f$  is the inclusion of a submanifold  $M$  into  $N$ , we use the notation  $\Omega^\bullet(N, M)$  and  $H^\bullet(N, M)$  for the relative cochain complex and relative cohomology associated to  $f$ , respectively.

**Proposition C.2.** Let  $f : M \rightarrow N$  be a map of manifolds with corners. Then there is a long exact sequence:

$$\dots \xrightarrow{f^*} H^p(M) \xrightarrow{\tilde{\iota}} H^{p+1}(f) \xrightarrow{\tilde{\pi}} H^{p+1}(N) \xrightarrow{f^*} H^{p+1}(M) \longrightarrow \dots \quad (\text{C.1})$$

where  $\tilde{\iota} : H^p(M) \rightarrow H^{p+1}(f)$  is the map  $\tilde{\iota}([\alpha]) := [(\alpha, 0)]$  and  $\tilde{\pi} : H^p(f) \rightarrow H^p(N)$  is the map  $\tilde{\pi}([\alpha, \beta]) := [\beta]$ .

*Proof.* Let  $\tilde{\Omega}^\bullet(M)$  be the “shifted negative de Rham complex” for  $M$ ; namely, the cochain complex with  $\tilde{\Omega}^k(M) := \Omega^{k-1}(M)$  and with differentials  $-d$  (for  $d$  the normal exterior differential on forms). Then clearly the collection of inclusions  $\iota_p : \tilde{\Omega}^p(M) \rightarrow \Omega^p(f)$  with  $i(\beta) := (0, \beta)$  for each  $p$  defines a map of cochain complexes  $\iota : \tilde{\Omega}^\bullet(M) \rightarrow \Omega^\bullet(f)$ .

On the other hand, let  $\pi_p : \Omega^p(f) \rightarrow \Omega^p(N)$  be the collection of projections  $\pi_p(\alpha, \beta) := \alpha$  for each  $p$ . Then we also have a chain map  $\pi : \Omega^\bullet(f) \rightarrow \Omega^\bullet(N)$ .  $\iota$  and  $\pi$  give rise to a short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{\Omega}^\bullet(M) \xrightarrow{\iota} \Omega^\bullet(f) \xrightarrow{\pi} \Omega^\bullet(N) \longrightarrow 0$$

Therefore, we have a long exact sequence of cohomology groups:

$$\dots \longrightarrow \tilde{H}^p(M) \xrightarrow{\tilde{\iota}} H^p(f) \xrightarrow{\tilde{\pi}} H^p(N) \longrightarrow \tilde{H}^{p+1}(M) \xrightarrow{\tilde{\iota}} \dots \quad (\text{C.2})$$

for  $\tilde{H}^\bullet(M)$  the cohomology of  $\tilde{\Omega}^\bullet(M)$ .

Note now that it is more or less obvious that  $\tilde{H}^p(M) = H^{p-1}(M)$  as vector spaces. To see that  $f^*$  is the connecting homomorphism for long exact sequence (C.2), note that, for  $\gamma \in \Omega^p(N)$  a closed form,  $d_f(\gamma, 0) = (d\gamma, f^*\gamma) = (0, f^*\gamma)$ . Therefore, with the identification  $\tilde{H}^p(M) = H^{p-1}(M)$ , (C.2) becomes (C.1).  $\square$

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